High Frequency Trading: Strategic Competition Between Slow and Fast Traders.

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7 march 2017

Abstract

In the following paper we analyze the strategic competition between fast and slow traders. The model of Kyle (1985) is adapted to analyze the effect of speed in such a model. A High Frequency Trader (HFT) is defined as a trader that has the ability to react to information faster than other informed traders and as a consequence can trade more than other traders. This trader benefits from low latency compared to slower trader. In such a setting, we prove the existence and the unicity of an equilibrium with fast and slow traders. We find that the speed advantage of High Frequency Traders (HFTs) has a beneficial effect on market liquidity as well as price efficiency. The positive effect on liquidity is present only if there are 2 or more HFTs. However, despite those effects slower traders are at a disadvantage as they are not able to trade on their private information as many times as their HFTs counterpart . When they are able to trade on it, most of their private information has been incorporated into prices due to the latency of HFTs. This implies that slower traders are worse off when HFTs are present. The speed differential benefits HFTs as they earn higher expected profits than their slower counterparts and also benefits liquidity traders. We find the existence of an optimal level of speed for HFT.

Keywords: High frequency trading, Insider, Volatility, Market efficiency.

JEL Classification: D43, D82, G14, G24.

1 Introduction

The last two decades have seen the explosion of computerized trading. High Frequency Trading (HFT) is only one aspect of computerized or algorithmic trading.[1] A definition of HFT is quite complex and can be given by describing its properties such as proprietary trading, very short holding periods, submission of a large number of orders that are rapidly cancelled, flat position at the end of the trading day, low margin per trade and the use of co-location

services (see Gomber et al. (2011)). According to the literature focusing on the US markets, between 40% and 70% of the trading volume in the US equity markets stems directly from HFT (see Biais and Woolley (2011)). The European and Asian-Pacific markets are slightly less exposed to HFT as 38% (for the European markets) and between 10%-30% (for the Asian-Pacific markets) of the traded volume is attributed to HFT. This phenomenon has initially been concentrated in equity markets. However, it has expanded beyond equity markets to other markets and to other asset classes such as fixed income markets, FX markets and futures markets.[2] This has been a result of the intense competition between HFTs on the equity markets and the desire to maintain a certain level of profits. HFT is now a feature of many markets. Some researchers see it as a permanent phenomenon with a temporary effect. In the same way as the introduction of telegraph, telephone and then computers gave a speed advantage to its early adopters that then disappeared as more and more traders adopted the new technology. Overall, the profit of HFTs is declining as a result of more and more HFTs being active in the different markets. However, due to its growth and presence in many markets, researchers have become more interested in HFT and have tried to assess its impact on markets. According to O'Hara (2015) more research both empirical and theoretical on HFT is still needed. This relatively new phenomenon (Algorithmic Trading) has also been the focus of the popular business press with an overwhelmingly negative view (see for instance Lewis (2014), Baer and Patterson (2014) and Lopez (2014)).

HFT offers different challenges such as measuring it and then assessing its impact on financial markets. When quantifying HFT the lack of a unique workable empirical definition proves to be problematic and two different approaches are used: a direct and an indirect one. None of the two approaches can correctly measure the activity stemming from HFT and this leads to different activity measures. The direct approach identifies HFT firms by their primary business and the types of algorithms they use whereas the indirect approach is based on the lifetime and the number of orders. The former possibly leads to a lower bound of the measure as some other

[1] As algorithmic trading (AT) is still a relatively new phenomenon a definition is slowly emerging. Prix et al. (2007) describes it as computerized trading controlled by algorithms without any human interventions. A more precise definition is given by Kirilenko and Lo (2013) as being "the use of mathematical models, computers, and telecommunication networks to automate the buying and selling of financial securities".

[2] Increased turnover in FX market has been found (increase of $657 billion from April 2007 to April 2010) and HFT has been indirectly linked to that increase (see the BIS Triennal Survey).

financial institutions may engage in HFT and the latter an upper bound. Using the two above approaches, Bouveret et al. (2014) finds that between 24% and 76% of the activity is linked to HFT. The research studies 100 stocks from nine European countries.

In the present paper, we analyze HFT in a theoretical model. Our definition of a fast trader (HFT) is straightforward and refers to a trader that can react to information faster than other informed traders and as a consequence can trade more than other traders. This trader benefits from low latency where low latency refers to the time it takes a trader to reacts to new information. Comparatively, a slow trader receives private and public information but needs time to process information and then to trade on it. Once the slow trader trades the fast trader has potentially traded several times (the number of times depends on the speed) and the slow trader is unable to trade on the information revealed by the HFT. We capture that difference between HFT and slow traders. The model analyzed is based on Kyle (1985) and we modify this model to allow for different speeds for traders. As a consequence, we are able to analyze the effect of speed in a Kyle (1985) framework. Following empirical findings, we assume that HFTs are informed (see Biais and Foucault (2014) and Biais et al. (2015) for instance). Existing models based on Kyle (1985) do not study the effect of the speed differential between traders (see Boco et al. (2016), Didri and Germain (2009), Foster and Viswanathan (1993) and finally Holden and Subrahmanyam (1992) among others). This allows us to look at the effect of changing speed and therefore different speed technologies onto different market measures.

The critical aspect for HFTs to realize gains and therefore keep their comparative advantage is to be able to trade fast and achieve low latency. This is obtained by substantial investment in infrastructure and also by the co-location of HFT's computers at the exchange. As an example, Spread Networks is reported to have spent $350 million to connect Wall Street and Chicago with a fibre optic cable in order to reduce latency by 3 milliseconds. Even such a small reduction in the latency is worth several hundred million of dollars. Co-location allows HFT firms to locate their servers close to the exchanges' servers decreasing the time to access market data. The cost of a co-located server varies from $7,000/month to around three times that amount depending on whether the direct exchange feed is added or not (see Ding et al. (2014)). The TABB group estimates that, for 2013, $1.5 billion has been invested in fast trading technologies. Some few papers have looked at that investment issue. Biais et al. (2015) find that because fast trading firms do not internalize the adverse selection costs they generate on slower trading firms, they overinvest in fast trading technologies. This overinvestment result also occurs in Pagnota and Philippon (2015) and Budish at al. (2014). The investment in fast trading technology is beyond the scope of our paper. However, our model shows that there is an optimal relative speed for fast traders. This optimal level naturally varies with both the number of fast and slow traders. It increases with the number of slow traders and varies non-monotonically with the number of fast traders.

Once a certain level of latency has been put in place through some investment, HFTs use strategies to benefit from certain market conditions. The majority of HFT strategies are designed to profit from high liquidity and low volatility in the market. However, the strategies used by HFTs are heterogeneous and can be divided in two categories referred to as market-making strategies i.e. liquidity-providing strategies and opportunistic strategies i.e. statistical arbitrage strategies. However there is a concern that as HFT are not market makers and have no obligation to provide liquidity, they may strategically provide liquidity and therefore may not supply it when most needed.[3] Some of the focus of the literature has been to analyze the impact of the former strategies. Hagstromer and Norden (2013) find that most HFTs on the Nasdaq-OMX Stockholm use market-making strategies and alleviate intraday price volatility. Menkveld (2013) specifically focuses on one HFT market maker and finds that this HFT, broadly speaking, behaves as a market maker managing his inventory position. The rest of the empirical literature overwhelmingly shows that the presence of HFT has increased market quality (increased liquidity) by decreasing bid-ask spreads and contributing to price efficiency (see Brogaard et al. (2014), Hendershott et al. (2011), Hasbrouck and Saar (2013), Brogaard (2010) and Menkveld (2014) to name but a few). Chabaud et al. (2014) also obtain that HFTs improve market efficiency by increasing liquidity and decreasing short term volatility. We confirm the finding on liquidity as we show that as the relative speed of the HFT increases, it augments the level of liquidity in the market. This has then a beneficial impact on liquidity traders as this increased liquidity leads to a reduction of their trading costs. However we prove that this result hinges on the presence of more than one HFT. Menkveld and Zoican (2016) shows that HFT may have a detrimental effect on the provision of liquidity and may reduce it. In their model, HFTs by being able to update their information faster and being able to trade on it reduce adverse selection in the market having a beneficial effect on liquidity. However, if their orders meet with High Frequency speculators, this may actually decrease liquidity. Whether the liquidity is positively or negatively affected depends on the security news to liquidity trader ratio. Further recent studies highlight the potential negative effect of the presence of HFTs such as the adverse selection effects brought by their presence (see for instance Biais at al. (2015) as explained above, see also Brogaard et al. (2014)). Cartea and Penalva (2012) show that liquidity traders pay higher prices when buying and sell at lower prices when HFTs are present. Jain et al. (2016) show that the introduction of Arrowhead high-speed-trading platform on the Tokyo Stock Exchange, enabling high frequency trading, increases the exposure to systemic risk.

The profit obtained by HFT strategies has also been under scrutiny. HFTs benefit by arbitraging prices and taking advantage of the difference in liquidity between distinct venues and have therefore gained from fragmented markets. HFTs earn a small amount of profit per

[3]As an example of this strategic supply of liquidity, several HFTs ceased to provide liquidity during the Flash Crash of May 2010. Kirilenko et al. (2016) conclude that HFTs did not trigger the Flash Crash but contibuted to it due to their repsonse to the selling pressure.

trade, however given the number of trades they conduct per day their profit can be extremely large. Evidences have suggested a decline in the profitability of HFT. This may be the result of more competition and/or the result of the increased cost of fast trading.[4] We find that the expected profit of HFT initially increases with their relative speed. However a large relative speed leads to lower expected profit. This suggests an optimal level of speed. The effect of the HFT's relative speed onto slower traders is clearer. Both the number of HFTs and their relative speed have a negative impact on slower traders.

In summary, we prove the existence of a unique equilibrium with fast and slow traders. We show that the presence of more than one HFT has a beneficial effect on liquidity and this benefits both liquidity traders and slow traders. However through the fact that slow traders trade on information and do not have the technology to react as fast as HFT, they are negatively affected. Because of their relatively slow speed, slow traders are limited in the use of their private information and this leads to slow traders being harmed by the presence of faster traders. This is captured by the fact that their expected profits are decreasing with the HFT's relative speed. and the number of HFTs. This has important regulatory implications. If the regulator focuses on small investors with no information, then the presence of HFT is beneficial. As those traders benefit from higher liquidity leading to lower prices. However, if we look at the effect on more sophisticated traders still with no or very limited resources to invest in speed, the presence of HFT has a negative impact on these traders.

The remainder of the paper is organized as follows. In Section 2, we present the model with fast and slow traders. In Section 3, we derive the equilibrium and show that it is unique for the benchmark case of one slow trader and one fast trader. We analyze the properties of the liquidity, price informativeness and expected profits in that setup. In Section 4, we look at the general case where several fast traders compete with several slow traders. In this setup, we characterize the linear equilibrium (existence and uniqueness) and we analyze among other things how the different market performance measures are affected by the HFT's speed. Finally, in Section 5, we make some concluding remarks. All proofs are gathered in the Appendix.

2 The Model

We first present the model in the Kyle (1985) parlance and we then re-interpret the model to a setting with HFT and therefore different trading speeds.

We consider a risky security which is traded during N sequential auctions in a time interval which begins at $t = 0$ and ends at $t = 1$. Let Δt_n be the time interval between the nth auction and the previous auction, we assume that $\Delta t_n = \frac{1}{N}$. At $t = 1$, the liquidation value of the asset

[4] See the Financial Times, February 13, 2013 and the New York Times, October 14, 2012 for evidences of both.

is revealed. It is denoted by $\tilde{v}$, with $\tilde{v} \sim N(\bar{v}, \sigma_v^2)$. For simplicity and without loss of generality, we assume $\bar{v} = 0$. We consider that we have two types of informed traders some that have invested in a technology permitting high frequency trading and others that have not. Although interesting, we do not model this investment decision and leave it for future research.

The fast and slow traders are defined as follows

- M_1 fast insiders (HFTs). At $t = 0$ they know the liquidation value perfectly. Each fast insider j submits an order ΔY_{jn} at each auction n, for $n = 1, \ldots, N$ and for $j = 1, \ldots, M_1$. The fast trader can react to information faster than other informed traders and as a consequence trades more than slower traders. This trader benefits from low latency where low latency refers to the time it takes a trader to react to new information.

- M_2 slow insiders. At time $t = 0$, they observe the liquidation value but they participate only to the last auction at time $t = 1$. Each slow insider i, for $i = 1, \ldots, M_2$ submits his unique order, denoted by ΔX_{iN} at the final auction N. This set up models the fact that the slow trader needs relatively more time to process his information and then to trade on it. With the aim of greatly simplifying the model we assume that they can only trade at the last auction.

The other two types of agents present in the market are now described

- Liquidity traders. They submit orders at each auction and do not possess any information about the fundamental value of the risky asset. We denote by $\Delta\tilde{u}_n$ their aggregate orders and we assume that $\Delta\tilde{u}_n$ are independently and identically normally distributed with zero mean and variance $\sigma_u^2\Delta t$. Also, we assume that $\Delta\tilde{u}_n$ are independent of $\tilde{v}$

- Competitive risk-neutral market makers. As in Foucault et al. (2016), market makers continuously price the asset and set the price p_n, for each trade n (or each auction n) in a Bayesian way (as in Kyle (1985)).

In the "typical" Kyle (1985), the number of times the asset can be traded between $t = 0$ and $t = 1$ is determined by the number of auctions. In our model it is determined by the speed of the different traders. It is assumed that the slow trader can only trade the asset once, whereas the HFT can trade it several times due to its speed. As a consequence, the number of auctions N can also be interpreted as the relative speed of the HFT or the HFT's speed advantage. In other words, N can be understood as how many more times the HFTs can trade relative to the slow traders. In that spirit, Δt_n can be interpreted as the time interval between the nth HFT's trade and the previous one, we assume that $\Delta t_n = \frac{1}{N}$ where N is the number of times the HFT can trade while the slow traders trade once.

The two types of informed market participants are strategic. At each auction n, the fast traders determine their optimal trading strategy by a process of backward induction in order to maximize their expected profits from their last trade N to the current trade, the nth trade.

We look for a linear equilibrium where each informed trader chooses an order which is linear in his private information and the previous public price.

Competition in market making drives the market makers' expected profits to zero, conditional on the aggregate submitted orders $\tilde{w_n}$. We also look for linear strategies for the market makers.

In the next sections we provide the main results of our paper namely the proposition stating the existence and uniqueness of the equilibrium for the two case scenario under study: one HFT competing with one slow trader, and several HFTs competing with several slow traders. However, in all scenario liquidity traders are present. We first look at the benchmark case with one fast and one slow trader.

3 One Fast Trader and One Slow Trader

We now look for the Bayesian Nash Equilibrium with one fast informed trader facing a unique slow insider.

3.1 The Equilibrium

In this section, we look for a linear equilibrium in which one HFT faces with one slow informed trader. We denote by ΔY_n the demand of the fast informed trader for his nth trade, for $n = 1, \ldots, N$ and we denote by ΔX_N, the order submitted by the slow insider.

Competitive risk-neutral market makers continuously set the linear price. The aggregate order flow is given by

$$\begin{cases} \tilde{w}_N = \Delta Y_N + \Delta X_N + \Delta \tilde{u}_N, \\ \tilde{w}_n = \Delta Y_n + \Delta \tilde{u}_n \qquad\qquad for\ n < N. \end{cases}$$

It should be pointed out that when the slow trader trades the fast trader has traded several times. However, the slow trader only observes p_0 and the future liquidation value he received as private information before trading. This is due to the slow relative speed assumption.

The next proposition gives the form of the equilibrium.

Proposition 3.1 *There exists a unique linear equilibrium in which the demand functions of* *both informed traders (HFT and slow trader) for each trade are:*

$$\begin{cases} \Delta X_n &= 0 \qquad for\ n < N, \\ \Delta X_N &= \beta_N^X(\tilde{v} - p_0), \end{cases} \tag{3.1}$$

$$\Delta Y_n = \beta_n^Y(\tilde{v} - p_{n-1})\Delta t_n. \tag{3.2}$$

The linear price, the error variance of prices and the expected profits are given respectively by:

$$\Delta p_n = P_n - p_{n-1} = \lambda_n \tilde{w_n}, \tag{3.3}$$

$$\Sigma_n = var(\tilde{v}|\tilde{w_1}, \ldots, \tilde{w_n}), \tag{3.4}$$

$$E[\pi_n^Y | p_1, \ldots, p_{n-1}, \tilde{v}] = \alpha_{n-1}^Y(\tilde{v} - p_{n-1})^2 + \delta_{n-1}^Y. \tag{3.5}$$

For $n < N$ the different coefficients are given as follows:

$$\alpha_{n-1}^Y = \frac{1}{4\lambda_n(1 - \lambda_n\alpha_n^Y)}, \tag{3.6}$$

$$\delta_{n-1}^Y = \delta_n^Y + \alpha_n^Y\lambda_n^2\sigma_u^2\Delta t_n, \tag{3.7}$$

$$\beta_n^Y\Delta t_n = \frac{1 - 2\lambda_n\alpha_n^Y}{2\lambda_n(1 - \lambda_n\alpha_n^Y)}, \tag{3.8}$$

$$\lambda_n = \frac{\beta_n^Y\Sigma_n}{\sigma_u^2}, \tag{3.9}$$

$$\Sigma_n = \Sigma_{n-1}(1 - \lambda_n\beta_n^Y\Delta t_n). \tag{3.10}$$

The boundary conditions at the last trade N are:

$$\alpha_{N-1}^Y = \frac{1}{9\lambda_N},\ \delta_{N-1}^Y = 0,\ \beta_N^Y\Delta t_N = \frac{1}{3\lambda_N}, \tag{3.11}$$

$$\lambda_N = \frac{2\beta_N^Y\Sigma_N}{\sigma_u^2},\ \Sigma_N = \Sigma_{N-1}(1 - 2\lambda_N\beta_N^Y\Delta t_N), \tag{3.12}$$

$$\begin{cases} \alpha_N^Y &= 0, \\ \delta_N^Y &= 0. \end{cases} \tag{3.13}$$

Proof: See Appendix.

After having established the existence, uniqueness and the equations of the equilibrium for our benchmark, we now turn to how the main performance measures of the model are affected by the presence of one HFT. We look at the effect of speed on the liquidity, informativeness and, finally, on expected profits of both HFTs and slow traders.

3.2 Liquidity

The liquidity parameter measures the adverse selection problem, in other words, the informational content of the order flow.

Numerical result 1: Liquidity

1. *Liquidity increases as a function of time however at an increasing rate.*
2. *Liquidity decreases with the relative speed or latency of the fast trader.*

The first point in result 1 shows how the HFT exploits his information. He gradually uses his information so that his information is not incorporated into prices too early. As he gets closer to the end of the trading day he trades more on his private information.

The second point states that the adverse selection problem increases with the speed of the fast trader. In that case, the HFT being a monopolistic trader fully exploits his speed advantage. This can be understood by looking at the graph of how the HFT exploits his private information (β_n^Y). As can be seen and as explained above, the trader gradually trades on his private information. Moreover, as the trader enjoys more speed the more intensely he trades on his private information at the later stage of the trading day. This result contradicts most of the results on the effect of speed on liquidity that show that liquidity increases with speed (see Brogaard et al. (2014), Hendershott et al. (2011), Hasbrouck and Saar (2013), Brogaard (2010) and Menkveld (2014)). However, a recent theoretical paper by Menkveld and Zoican (2016) shows that the above result may be changed depending on the security news to liquidity trader ratio. Our result is due to the assumption that only one trader has access to a technology giving a relative speed advantage.

The above result can be seen in Figure 1 of the Appendix. Figure 2 shows how the HFT gradually trades on information and accelerates his intensity towards the end of the trading day.

3.3 Informativeness and Volatility

Numerical result 2: Price Informativeness

1. *Price informativeness* $\left(\frac{1}{\Sigma_n}\right)$ *increases as a function of time.*
2. *The effect of the HFT's relative speed is non-monotonic.*

As explained above, the fast trader has a monopolistic position for $N-1$ of his orders among his N orders. He gradually trades on his long-lived private information which is, in turn, gradually incorporated into prices. As a consequence price efficiency is increased.

It can be seen that the effect of speed on price efficiency is not monotonic. Higher relative speed implies that markets are less informationally efficient early on, and eventually reveal more information closer to the end of the trading day. Again this result depends on the fact that the HFT is monopolistic. This can be seen in Figure 3 of the Appendix.

Numerical result 3: Volatility

1. *Price volatility increases and then decreases as a function of time.*
2. *The effect on volatility of the HFT's relative speed is non-monotonic.*

The above two points can be seen in Figure 4. The effect of a single HFT on price volatility is not as clear cut. The presence of one HFT leads to a build up in volatility as he faces no competition. The competition with the slow trader leads to a decrease in volatility. An increase in the HFT's speed leads to more trade opportunities for the HFT however the effect of that increase on price volatility is non-monotonic.

3.4 Expected Profits

Numerical result 4: Expected Profits

1. *Provided $N > 2$, the expected profit of the fast trader increases with its speed, whereas the expected profit of the slow trader decreases with the speed of the HFT.*
2. *The fast trader always obtains higher expected profits than the slow trader.*

As previously commented upon, the HFT enjoys a monopolistic position and the greater its speed the more he can exploit that position. Not surprisingly, its expected profits are then increasing with its speed. The HFT's speed because it strengthens its monopolistic position has a detrimental effect on the slow trader. Once the slow trader can trade most of his private information which is shared with the HFT has been incorporated into prices. As the speed on the HFT increases more of the private information is revealed in prices and the less scope the slow trader can benefit from his private information. This then leads to decreasing expected profits of the slow trader with the speed of the HFT and to the slow trader's expected profit being lower than the fast trader's. In that case, higher relative speed only benefits HFTs. Indeed, the decrease in liquidity due to the increase in relative speed of the HFT makes all other market participants worse off (apart from the market makers as their expected profits are equal to zero). This result makes a stronger case for the regulation of high frequency trading.

The above statements can be seen in Figures 17 and 19 of the Appendix. The reader can refer to the curve where $M_1 = M_2 = 1$.

4 Several Fast and Slow Traders

We now look at the more general case where $M_1 \geq 1$ several fast traders compete between each other as well as compete against $M_2 \geq 1$ slow traders.

4.1 The Equilibrium

Similarly to the previous section, we denote by ΔY_{jn} the demand of the jth fast informed trader for the nth order, for $j = 1, \dots, M_1$ and for $n = 1, \dots, N$. The aggregate nth orders stemming from the fast insiders are denoted by $\sum_{j=1}^{M_1} \Delta Y_{j_n} = \Delta Y_n$. We denote by ΔX_{iN}, the order submitted by the ith slow insider for $i = 1, \dots, M_2$. The aggregate orders from slow insiders are denoted by $\sum_{i=1}^{M_2} \Delta X_{i_N} = \Delta X_N$.

The market makers behave as before. The aggregate order flow is given by

$$\begin{cases} \tilde{w}_N = \sum_{j=1}^{M_1} \Delta Y_{j_N} + \sum_{i=1}^{M_2} \Delta X_{i_N} + \Delta \tilde{u}_N = \Delta Y_N + \Delta X_N + \Delta \tilde{u}_N, \\ \tilde{w}_n = \sum_{j=1}^{M_1} \Delta Y_{j_n} + \Delta \tilde{u}_n = \Delta Y_n + \Delta \tilde{u}_n \qquad\qquad for\ n < N. \end{cases}$$

Using a symmetry argument, it is straightforward to show that, at the equilibrium, all informed traders of the same type have an identical strategy. Also, the demand of the ith slow participant is $\Delta X_{i_N} = \beta^X_{i_N}(\tilde{v} - p_0) = \beta^X_N(\tilde{v} - p_0)$ and the demand for the nth order of the jth fast insider is $\Delta Y_{jn} = \beta^Y_{jn}(\tilde{v} - p_{n-1})\Delta t_n = \Delta Y_n = \beta^Y_n(\tilde{v} - p_{n-1})\Delta t_n$.

As before, although the slow traders trade at the last auction they are trading on the knowledge of p_0 and their private information.

The following proposition states the linear equilibrium.

Proposition 4.2 *There exists a unique linear equilibrium such that*

The demands by strategic traders are given by

$$\begin{cases} \Delta X_n = 0 \qquad pour\ n < N, \\ \Delta X_N = M_2 \beta^X_N(\tilde{v} - p_0), \\ \Delta Y_n = M_1 \beta^Y_n(\tilde{v} - p_{n-1})\Delta t_n. \end{cases} \tag{4.14}$$

The price is given by

$$\Delta p_n = \lambda_n \tilde{w}_n. \tag{4.15}$$

We then have the following

$$\Sigma_n = \mathrm{var}(\tilde{\mathrm{v}}|\tilde{w}_1, \ldots, \tilde{w}_n), \tag{4.16}$$

$$E[\pi_{\mathrm{n}}^{\mathrm{Y}}|p_1, ..., p_{n-1}, \tilde{v}] = \alpha_{n-1}^Y(\tilde{v} - p_{n-1})^2 + \delta_{n-1}^Y, \tag{4.17}$$

$$\alpha_{n-1}^Y = \frac{1 - \lambda_n \alpha_n^Y}{\lambda_n(M_1(1 - 2\lambda_n \alpha_n^Y) + 1)^2}, \tag{4.18}$$

$$\delta_{n-1}^Y = \delta_n^Y + \alpha_n^Y \lambda_n^2 \sigma_u^2 \Delta t_n, \tag{4.19}$$

$$\beta_n^Y \Delta t_n = \frac{1 - 2\lambda_n \alpha_n^Y}{\lambda_n(M_1(1 - 2\lambda_n \alpha_n^Y) + 1)}, \tag{4.20}$$

$$\lambda_n = \frac{M_1 \beta_n^Y \Sigma_n}{\sigma_u^2}, \tag{4.21}$$

$$\Sigma_n = \Sigma_{n-1}(1 - M_1 \lambda_n \beta_n^Y \Delta t_n). \tag{4.22}$$

The boundary conditions are given by:

$$\alpha_{N-1}^Y = \frac{1}{(M_1 + M_2 + 1)^2 \lambda_N}, \ \delta_{N-1}^Y = 0, \tag{4.23}$$

$$\beta_N^Y \Delta t_N = \frac{1}{(M_1 + M_2 + 1)\lambda_N}, \ \lambda_N = \frac{(M_1 + M_2)\beta_N^Y \Sigma_N}{\sigma_u^2}, \tag{4.24}$$

$$\Sigma_N = \Sigma_{N-1}(1 - (M_1 + M_2)\lambda_N \beta_N^Y \Delta t_N), \tag{4.25}$$

$$\begin{cases} \alpha_N^Y &= 0, \\ \delta_N^Y &= 0. \end{cases} \tag{4.26}$$

Proof: See Appendix.

In what follows, we focus on the properties of our general model in terms of liquidity, informativeness and profits during the two phases of our trading game: a phase (early auctions until the penultimate one) where only the fast trader is active as opposed to the phase (last auction) where both insiders are active.

4.2 Liquidity

Numerical result 5: Liquidity

1. *Liquidity ($\frac{1}{\lambda}$) increases over time.*
2. *Liquidity increases with the speed of the fast trader.*
3. *The effect of the number of HFTs depends on the number of HFTs. If there are more than 1 HFTs, increasing their number will increase liquidity. The effect of the slow traders is not as clear.*

Ceteris paribus, we obtain that liquidity increases over time and this can be seen in all the figures representing the liquidity (from Figures 5 to 8 in the Appendix).

The second point states that the speed of the HFTs is beneficial to market quality as more speed increases the liquidity of the market. Higher relative speed leads to more competition and this benefits the level of liquidity. This can be seen in Figure 5 of the Appendix.

In such a model most of the competition comes from the early HFTs trades. The above result tells us that the more HFTs compete, the better the level of liquidity (this can be seen in Figure 6). More competition leads to a reduction of the adverse selection problem.

The two last results, described above, echo the overwhelming finding in the literature that the presence of HFTs increases liquidity in markets by decreasing bid-ask spreads (see Brogaard et al. (2014), Hendershott et al. (2011), Hasbrouck and Saar (2013), Brogaard (2010) and Menkveld (2014)). However, a recent theoretical paper by Menkveld and Zoican (2016) shows that the above result maybe changed depending on the security news to liquidity trader ratio.

The competition of the HFT against other HFTs always increase their reaction to private information and this does not depend on the number of slow traders. This increase competition can either be due to an increase in their number or an increase of their relative speed advantage. This can be seen in Figures 9, 11 and 12. The competition of HFTs against the slow traders obviously depends on the number of HFTs. If there is one HFT, that trader gradually increases its response to private information. However, when competing against more than one slow trader for the last trade, he strategically reduces his intensity to reduce the impact of the aggregate order flow on the price. This reduction does not happen when there are more than one HFT. This can be seen in Figure from 9 to 12.

Interestingly, it can be seen from Figure 6, that the liquidity for early trades is not monotonic with the number of fast traders. It initially decreases with the number leading to the fact that a market with a single HFT is more liquid early on than any other markets with, given our parameters configuration, a number of competing HFTs between 2 and 9. This can be explained by the strategic behavior of the HFTs trying to “smooth” their information revelation by gradually trading on their private information.

4.3 Informativeness and Volatility

Numerical result 6: Price Informativeness

1. *Price informativeness* $\left(\frac{1}{\Sigma_n}\right)$ *increases with time.*

2. *Price informativeness also increases with the number of fast traders and their relative speed.*

This result shows that the competition between fast traders leads them to reveal their information at the earlier auctions. Therefore, most of the informativeness of prices is provided by the fast traders as when the slow traders trade most of their private information has been revealed. The two points can be seen in Figure 13 in the Appendix.

As can be seen from Figure 14, the effect of slow traders on price efficiency is very small. This is due tot he fact that once their orders reach the market most of their private information has been already incorporated into prices.

Numerical result 7: Volatility

1. *The evolution of price volatility over time depends on the number of slow traders, the number of HFTs and their relative speed.*
2. *Price volatility may be decreasing or increasing with the HFTs' speed.*
3. *Price volatility increases with the number of slow traders whereas it is non-monotonic with the number of HFTs.*

An interesting parameter is price volatility however, the effect of HFT on price volatility is not clear.

The above statements can be seen Figures 15 and 16.

4.4 Expected Profits

Numerical result 8: Effect of the Number of Traders on Expected Profits

1. *Effect of HFTs: An increase in the number of HFTs leads to lower aggregate expected profits for slow traders. If the number of HFTs is low and their speed advantage is low enough, an increase in the number of HFTs increases aggregate profits for the HFTs. In other words, when HFTs face a low competitive environment, be it relative speed or number of HFTs, their aggregate profits increase with* M_1.
2. *Effect of slow traders: An increase in the number of slow traders leads to lower aggregate profits for HFTs. For a low number of slow traders, an increase in the number of slow traders increases the aggregate profits of slow traders and this independently of the speed advantage of the HFTs. For a large number of slow traders, it decreases the aggregate profits of slow traders.*

In other words, slow traders are negatively affected by the presence of HFTs. The more HFTs the worse off they are. The competition between HFTs leads to most of the slow traders' private information to be revealed before they have the chance to trade on their private information. This can be seen in Figure 22.

When there are two or more fast traders and their relative speed is high enough, competition between HFTs decreases their aggregate expected profits. Figures 17 and 18 illustrate that point.

Competition from slow traders decrease the expected profits from HFTs. This is shown in Figures 19, 20 and 21.

Figures 22 illustrates the effect of the number of slow traders on the the expected profits of the slow traders. They show that the aggregate expected profits are non monotonic with M_2, the number of slow traders.

Numerical result 9: Effect of Relative Speed on Expected Profits

1. *The aggregate expected profits of the fast traders initially increase with their latency and then decrease with it. This leads to the existence of an optimal level of latency.*
2. *The aggregate expected profits of the slow traders always decrease with the HFTs relative speed.*
3. *HFTs obtain larger expected profits than slow traders.*

This last numerical result highlights the relationship of the expected profit of both the fast traders and the slow traders with the relative speed or latency of the HFTs. This means that it links the profit with the investment in the fast technology. The fast technology can either be locating servers on the exchange and/or investing in fibre optic etc... Our result then states that investing in fast technology will benefit the few informed traders able to invest in it and provided they do not invest too much in the technology. This is illustrated by Figures 17, 18, 19, 20, 21 and 22. If too much is invested, fast traders experience a decrease in their expected profits except when being a monopolistic trader. It is always the case that slow traders see their expected profit decrease with the investment in the fast technology despite the fact that liquidity is increased by higher relative speed (see Figure 22). They are then made worse off by the presence of HFTs. Liquidity traders, due to an increase in liquidity, have of their cost of trading reduced.

The last point above echoes Baer and Patterson (2014) stating that higher speed from some traders gives them an unfair advantage (see Figure 23).

Numerical result 8 and 9 may help us understand the recent findings that HFTs have seen their profit reduced. Given our results it may be due to more and more traders investing in fast technology and leading to more competition and/or to a suboptimal investment in fast technology.

5 Policy Implications

Comparing the two models can help us draw some policy implications.

In the benchmark, we find that liquidity decreases with the relative speed of HFTs whereas we obtain the opposite result when there are strictly more than one HFT. We also find that relative speed decreases price volatility. These observations can help with the regulation of HFTs. Looking at liquidity, any type of regulation that promotes competition between HFTs such as increasing their number will have a beneficial effect. This can be achieved in different ways. Some of the discussions have focused directly on the speed of HFTs and have proposed a speed limit to decrease their speed advantage. A speed limit is a proposition put forward by EBS, one of the two dominant platforms in the foreign exchange market.[5] This can be achieved in different ways. The proposition of EBS is to batch orders together and execute them in a random way. Another proposition from regulators in Australia and Europe is to impose resting periods. The discussion around the creation of the IEX stock market is also relevant and interesting. This market has been created as a response to the perception that speed gives an unfair advantage to the market participants who benefit from it. The IEX does not allow traders to co-locate their servers close to the market's servers. A delay of some fraction of a second is artificially added up to eliminate the speed advantage of some HF traders. Other propositions have been to implement a fee structure directed at HFTs. For instance, the Moscow Exchange is looking at implementing fees that would apply to traders using many small orders (this is a feature of HFTs). In China, a limit on the number of trades in Futures markets has been implemented. Traders can trade in the same instrument up to 500 times a day. This puts a significant limit in the number of trades HFTs can execute.

If we look at the effect of HFTs on price volatility and comparing Figures 4, 15 and 16 we can see that the presence of HFTs leads to changes in volatility. It appears that the effect of the speed and the number of HFTs is not very clear. However, comparing the different Figures on price volatility we can see that when we compare the benchmark case to the general case price volatility is non monotonic in the number of HFTs. Early price volatility is lower with one HFT whereas late price volatility is lower the more HFTs compete. Given the comparative statics we obtain, policy recommendations are difficult to make.

[5]See the article in the Financial Times from March 7, 2016 entitled *US exchanges: the "speed bump" battle.* See also another article from the Financial Times entitled *HF Traders face speed limit* from April 28, 2013. Finally, the New York Times Magazine from October 8, 2013has published the follwoing article *Putting a speed limit on the Stock Market.*

6 Conclusion

In the following paper we analyze the effect of HFT on markets. We define a HFT as being a trader that benefits from low latency and as such can trade on information much faster than a slow trader. We adapt the model from Kyle (1985) to allow traders to be able to trade at different speeds. We prove the existence and the unicity of the equilibrium with fast and slow traders. We derive a benchmark model where one HFT competes against one slow trader. We then generalize our results to the case where several HFTs compete against each other as well as against several slow traders.

We get the following results. In the benchmark, we obtain that the liquidity decreases with the relative speed of the HFTs. This leads to the fact that all other traders, except market makers, are made worse off by the presence of the HFT. We also obtain that the effect of the presence of the HFT on price volatility is not clear. For the general case, we prove that the higher speed from some traders improves liquidity and price efficiency. We also find that speed is beneficial to HFTs as higher speed leads to the fact that they earn higher expected profits than slower traders. Higher speed increases the scope to use their private information. Furthermore, we obtain that speed has a detrimental effect on slow traders. The faster HFTs can trade the lower the slower traders' expected profits. This happens despite the fact that liquidity increases with speed. This is due to the fact that the higher the speed of the HFTs the more they can trade on their private information leading to the fact that when slower traders can trade most of their private information has already been incorporated into prices. This echoes Baer and Patterson (2014) stating that higher speed from some traders gives them an unfair advantage. Finally, we obtain that the HFTs' expected profits are initially increasing with their speed advantage. This speed advantage dissipates for higher speed and their expected profits decrease with speed. This suggests an optimal level of latency. Overall price volatility is improved by the competition between HFTs and their relative speed.

Our results show that the improved liquidity (seen in the general case) will not benefit all market participants. An improved liquidity will reduce the losses by liquidity traders. Slower informed traders do not benefit from this improved liquidity as their expected profits decrease with the HFTs' latency.

Our paper also recommends more competition in HF trading as this may improve liquidity. Regarding price volatility, any policy recommendations are difficult to make.

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8 Appendix

Proof of Proposition 1

This is proved by setting $M_1 = M_2 = 1$ in Proposition 2 and the following the exact same steps as in Proposition 2.

Proof of the Proposition 2

We look for a linear equilibrium. The fast insiders determine for each of their orders the one that optimizes their expected profits given their conjectures about the both fast and slow traders' strategies.

The linear equilibrium implies that the price set for the nth order flow by the risk-neutral market makers is: $p_n = p_{n-1} + \lambda_n w_n$.

For $n < N$, the fast traders are the only informed market participants. We conjecture the linear strategy played by the jth fast trader for his nth order:

$$\Delta Y_{jn} = \beta^Y_{jn}(\tilde{v} - p_{n-1})\Delta t_n,$$

where $\tilde{v}$ is his private information (the liquidation value of the risky asset). Since all the insiders receive the same information at time $t = 0$, by using symmetric argument their strategies are identical at the equilibrium. Therefore, we suppress the "j" subscript from the reaction β_{jn} and the expected profit π_{jn} of the jth fast informed trader. One can then consider the profit of this jth fast informed trader which is realized for the nth order, and what remains to be gained from the next order to the end of trading. This is given below:

$$\begin{aligned} E[\pi^{\mathrm{Y}}_{\mathrm{n}}|p_1,\ldots,p_{n-1},\tilde{v}] &= \max_{\Delta Y_{jn}} \left(E[(\tilde{\mathrm{v}} - \mathrm{p_n})\Delta \mathrm{Y_{jn}}|p_1,\ldots,p_{n-1},\tilde{v}] + E[\pi^{\mathrm{Y}}_{\mathrm{n+1}}|p_1,...,p_{n-1},\tilde{v}] \right), \\ &= \max_{\Delta Y_{jn}}(I + II). \end{aligned}$$

with $I = E[(\tilde{\mathrm{v}} - \mathrm{p_n})\Delta \mathrm{Y_{jn}}|p_1,\ldots,p_{n-1},\tilde{v}]$ and $II = E[\pi^{\mathrm{Y}}_{\mathrm{n+1}}|p_1,...,p_{n-1},\tilde{v}]$.

We have

$$I = E\left[(\tilde{v} - (p_{n-1} + \lambda_n(\Delta Y_{jn} + \Delta Y^* + \Delta\tilde{u}_n)))\Delta Y_{jn}|p_0, \ldots, p_{n-1}, \tilde{v}\right],$$

where ΔY^* is the sum of the orders submitted at the same time by the $M_1 - 1$ other fast informed traders.

By considering that $\tilde{u}_n$ and $\tilde{v}$ are independent and that $E(\tilde{u}) = 0$, we obtain:

$$I = (\tilde{v} - p_{n-1})\Delta Y_{jn} - \lambda_n(\Delta Y_{jn})^2 - \lambda_n \Delta Y_{jn} \Delta Y^*.$$

On the other hand, we have:

$$II = E\left[\alpha_n^Y (\tilde{v} - p_n)^2 + \delta_n^Y |p_0, \ldots, p_{n-1}, \tilde{v}\right],$$

$$II = E\left[\alpha_n^Y (\tilde{v} - p_{n-1} - \lambda_n(\Delta Y_{jn} + \Delta Y^* + \Delta\tilde{u}_n))^2 + \delta_n^Y |p_0, \ldots, p_{n-1}, \tilde{v}\right].$$

This leads to:

$$\begin{aligned} II &= \alpha_n^Y (\tilde{v} - p_{n-1})^2 - 2\lambda_n \alpha_n^Y (\tilde{v} - p_{n-1})(\Delta Y_{jn} + \Delta Y^*) \\ &\quad + \lambda_n^2 \alpha_n^Y \left(\sigma_u^2 \Delta t_n + (\Delta Y_{jn})^2 + (\Delta Y^*)^2 + 2\Delta Y_{jn} \Delta Y^*\right) + \delta_n^Y. \end{aligned}$$

Considering the first order condition of the above maximization problem leads to:

$$(\tilde{v} - p_{n-1}) - 2\lambda_n \Delta Y_{jn} - \lambda_n \Delta Y^* - 2\lambda_n \alpha_n^Y (\tilde{v} - p_{n-1}) + 2\lambda_n^2 \alpha_n^Y \Delta Y_{jn} + 2\lambda_n^2 \alpha_n^Y \Delta Y^* = 0.$$

At the equilibrium, all insiders submit identical orders since they have received the same information leading to $\Delta Y^* = (M - 1)\Delta Y_{jn}$. Hence at the equilibrium we find:

$$\Delta Y_{jn} = \frac{1 - 2\lambda_n \alpha_n^Y}{\lambda_n \left[1 + M_1(1 - 2\lambda_n \alpha_n^Y)\right]} (\tilde{v} - p_{n-1}).$$

We then identify the reaction of the jth fast informed trader to his private information and to the previous price for his nth order:

$$\beta_n^Y \Delta t_n = \frac{1 - 2\lambda_n \alpha_n^Y}{\lambda_n \left[1 + M_1(1 - 2\lambda_n \alpha_n^Y)\right]}.$$

Finally, the second order condition yields to:

$$\lambda_n(1 - \lambda_n \alpha_n^Y) > 0. \tag{8.27}$$

On the other hand, the market efficiency condition implies that λ_n is the regression coefficient of $\tilde{v}$ on $\tilde{w}_n$ conditional on $\tilde{w}_1, \ldots, \tilde{w}_n$, in other words:

$$\lambda_n = \frac{cov(\tilde{v}, \tilde{w}_n)_{|\tilde{w}_1 \ldots, \tilde{w}_{n-1}}}{var(\tilde{w}_n)_{|\tilde{w}_1 \ldots, \tilde{w}_{n-1}}}.$$

By developing, we obtain:

$$\lambda_n = \frac{M_1 \beta_n^Y \Sigma_{n-1}}{M_1^2 (\beta_n^Y)^2 \Delta t_n \Sigma_{n-1} + \sigma_u^2}.$$

We now calculate the variance of error prices for the nth order Σ_n:

$$\Sigma_n = var(\tilde{v}|\tilde{w}_1, \ldots, \tilde{w}_n) = var(\tilde{v}|\tilde{w}_1, \ldots, \tilde{w}_{n-1}) - \frac{cov^2_{|\tilde{w}_1, \ldots, \tilde{w}_{n-1}}(\tilde{v}, \tilde{w}_n)}{var(\tilde{v}|\tilde{w}_1, \ldots, \tilde{w}_{n-1})}.$$

We derive the following expressions of Σ_n and λ_n respectively:

$$\Sigma_n = \frac{\Sigma_{n-1} \sigma_u^2}{M_1^2 (\beta_n^Y)^2 \Delta t_n \Sigma_{n-1} + \sigma_u^2},$$

$$\lambda_n = \frac{M_1 \beta_n^Y \Sigma_n}{\sigma_u^2},$$

$$\Sigma_n = \Sigma_{n-1}(1 - \lambda_n M_1 \beta_n^Y \Delta t_n).$$

Finally, for determining the relationship between α_n^Y and α_{n-1}^Y as well as between δ_n^Y and δ_{n-1}^Y we substitute the expression of ΔY_{jn} into the fast trader's expected profit. We then obtain:

$$\begin{aligned}
E[\pi_n^Y | p_0, \ldots, p_{n-1}, \tilde{v}] = &(\tilde{v} - p_{n-1}) \Delta Y_{jn} - \lambda_n (\Delta Y_{jn})^2 - \lambda_n \Delta Y_{jn} \Delta Y^* \\
&+ \alpha_n^Y (\tilde{v} - p_{n-1})^2 - 2\lambda_n \alpha_n^Y (\tilde{v} - p_{n-1})(\Delta Y_{jn} + \Delta Y^*) \\
&+ \lambda_n^2 \alpha_n^Y \left(\sigma_u^2 \Delta t_n + (\Delta Y_{jn})^2 + (\Delta Y^*)^2 + 2\Delta Y_{jn} \Delta Y^* \right) + \delta_n^Y, \\
= &\alpha_{n-1}^Y (\tilde{v} - p_{n-1})^2 + \delta_{n-1}^Y.
\end{aligned}$$

Thus, we have:

$$
\begin{aligned}
E[\pi_n^Y|p_0,\dots,p_{n-1},\tilde{v}] = & (\tilde{v}-p_{n-1})\beta_n^Y(\tilde{v}-p_{n-1})\Delta t_n - \lambda_n(\beta_n^Y(\tilde{v}-p_{n-1})\Delta t_n)^2 \\
& - \lambda_n\beta_n^Y(\tilde{v}-p_{n-1})\Delta t_n(M_1-1)\beta_n^Y\Delta t_n + \alpha_n^Y(\tilde{v}-p_{n-1})^2 \\
& - 2\lambda_n\alpha_n^Y(\tilde{v}-p_{n-1})(\beta_n^Y(\tilde{v}-p_{n-1})\Delta t_n + (M_1-1)\beta_n^Y\Delta t_n) \\
& + \lambda_n^2\alpha_n^Y \\
& + 2\beta_n^Y(\tilde{v}-p_{n-1})\Delta t_n(M_1-1)\beta_n^Y\Delta t_n \\
= & \alpha_{n-1}^Y(\tilde{v}-p_{n-1})^2 + \delta_{n-1}^Y,
\end{aligned}
$$

$$
\alpha_{n-1}^Y = \frac{1-\lambda_n\alpha_n^Y}{\lambda_n\left[M_1(1-2\lambda_n\alpha_n^Y)+1\right]^2},
$$

$$
\delta_{n-1}^Y = \delta_n^Y + \alpha_n^Y\lambda_n^2\sigma_u^2\Delta t_n. \tag{8.28}
$$

We now determine the demand of the insiders at the last auction $n = N$.

The ith slow informed trader chooses his demand ΔX_{iN} that maximizes his profit knowing his information that he receives at time $t = 0$, that is to say, his private signal $\tilde{v}$ and the public price p_0. Therefore his maximization problem is:

$$
E[\pi_N^X|p_0,\tilde{v}] = \max_{\Delta X_{iN}} E[\Delta X_{iN}(\tilde{v}-p_N)|p_0,\tilde{v}],
$$

with $p_N = \Delta X_{iN} + \Delta X^* + \Delta Y_N + \Delta\tilde{u}_N$; ΔX^* represents the aggregate orders submitted by the (M_2-1) other low informed traders and ΔY_N is the sum of the orders of the fast informed traders.

The first order condition implies that:

$$
(\tilde{v}-p_{N-1}) - \lambda_N\Delta Y_N - 2\lambda_N X_{iN} - \lambda_N\Delta X^* = 0.
$$

At the equilibrium the slow informed traders submit the same orders, in other words $\Delta X^* = (M_2-1)\Delta X_{iN}$. Hence the first order condition is given by:

$$
\Delta X_{iN} = \frac{1}{\lambda_N(M_2+1)}(\tilde{v}-p_{N-1}) - \frac{\Delta Y_N}{(M_2+1)}.
$$

The jth fast informed trader solves the following maximization problem:

$$
E[\pi_N^Y|p_0,\dots,p_{N-1},\tilde{v}] = \max_{\Delta Y_{jN}} E[\Delta Y_{jN}(\tilde{v}-p_N)|p_0,\dots,p_{N-1},\tilde{v}],
$$

$$E[\pi_N^Y|p_0,\dots,p_{N-1},\tilde{v}] =$$
$$\max_{\Delta Y_{jN}} E[\Delta Y_{jN}\left((\tilde{v}-p_{N-1})-\lambda_N(\Delta Y_{jN}+\Delta Y^*+\Delta X_N+\Delta\tilde{u}_n)\right)|p_0,\dots,p_{N-1},\tilde{v}],$$

with ΔY^* the aggregate orders submitted by the (M_1-1) other fast informed traders and ΔX_N the aggregate orders of the slow informed traders.

This leads to,

$$E[\pi_N^Y|p_0,\dots,p_{N-1},\tilde{v}] = \max_{\Delta Y_{jN}} \left(\Delta Y_{jN}(\tilde{v}-p_{N-1})-\lambda_N(\Delta Y_{jN})^2-\lambda_N\Delta Y_{jN}\Delta Y^*-\lambda_N\Delta Y_{jN}\Delta X_N\right).$$

The first order condition is given by:

$$(\tilde{v}-p_{N-1})-2\lambda_N\Delta Y_{jN}-\lambda_N\Delta Y^*-\lambda_N\Delta X_N=0.$$

At the equilibrium we have $\Delta Y^*=(M_1-1)\Delta Y_{jN}$. We also obtain the order of the jth fast informed trader:

$$\Delta Y_{jN}=\frac{1}{\lambda_N(M_1+1)}(\tilde{v}-p_{N-1})-\frac{\Delta X_N}{(M_1+1)}.$$

In sum, we have:

$$\left\{\begin{array}{rcl}\sum_{i=1}^{M_2}\Delta X_{iN} & = & \Delta X_N=\frac{M_2}{\lambda_N(M_2+1)}(\tilde{v}-p_{N-1})-\frac{M_2\Delta Y_N}{(M_2+1)},\\ \sum_{j=1}^{M_1}\Delta Y_{jN} & = & \Delta Y_N=\frac{M_1}{\lambda_N(M_1+1)}(\tilde{v}-p_{N-1})-\frac{M_1\Delta X_N}{(M_1+1)}.\end{array}\right.$$

This system of equations implies that:

$$\Delta X_{iN}=\Delta Y_{jN}=\frac{1}{\lambda_N(M_1+M_2+1)}(\tilde{v}-p_{N-1}).$$

On the other hand, the error variance of price at the final auction is:

$$\Sigma_N=var[\tilde{v}|w_1,\dots,w_{N-1},w_N]=\Sigma_{N-1}-\frac{cov^2(\tilde{v},w_N)_{|w_1,\dots,w_{N-1}}}{var(w_N)_{|w_1,\dots,w_{N-1}}}.$$

This leads to:

$$\Sigma_N = \frac{\sigma_u^2 \Delta t_N \Sigma_{N-1}}{M_1^2(\beta_N^Y \Delta t_N)^2 \Sigma_{N-1} + M_2^2(\beta_N^X)^2 \Sigma_{N-1} + \sigma_u^2 \Delta t_N}.$$

The liquidity parameter is given by:

$$\begin{aligned}
\lambda_N &= \frac{cov(\tilde{v}, w_N)_{|w_1,\ldots,w_{N-1}}}{var(w_N)_{|w_1,\ldots,w_{N-1}}}, \\
&= \frac{cov(\tilde{v}, M_1\beta_N^Y \Delta t_N(\tilde{v} - p_{N-1}) + M_2\beta_N^X(\tilde{v} - p_0) + \Delta\tilde{u}_N)_{|w_1,\ldots,w_{N-1}}}{var(M_1\beta_N^Y \Delta t_N(\tilde{v} - p_{N-1}) + M_2\beta_N^X(\tilde{v} - p_0) + \Delta\tilde{u}_N)_{|w_1,\ldots,w_{N-1}}}, \\
&= \frac{M_1\beta_N^Y \Delta t_N \Sigma_{N-1} + M_2\beta_N^X \Sigma_{N-1}}{M_1^2(\beta_N^Y \Delta t_N)^2 \Sigma_{N-1} + M_2^2(\beta_N^X)^2 \Sigma_{N-1} + \sigma_u^2 \Delta t_N}.
\end{aligned}$$

Since $\Delta X_{iN} = \Delta Y_{jN}$ for all $i = 1, \ldots, M_2$ and $j = 1, \ldots, M_1$, we have that $\beta_N^X = \beta_N^Y \Delta t_N$ and the following relationships:

$$\Sigma_N = \Sigma_{N-1}\left(1 - (M_1 + M_2)\lambda_N \beta_N^Y \Delta t_N\right),$$

and

$$\lambda_N = \frac{(M_1 + M_2)\beta_N^Y \Sigma_N}{\sigma_u^2}.$$

The boundary conditions give:

$$\begin{cases} \alpha_N^Y &= 0, \\ \delta_N^Y &= 0, \end{cases}$$

and

$$\beta_N^Y \Delta t_N = \beta_N^X = \frac{1}{\lambda_N(M_1 + M_2 + 1)}.$$

■

9 Figures

9.1 One HFT and one slow trader

All graphs are done with $\sigma_v^2 = \sigma_u^2 = 1$.

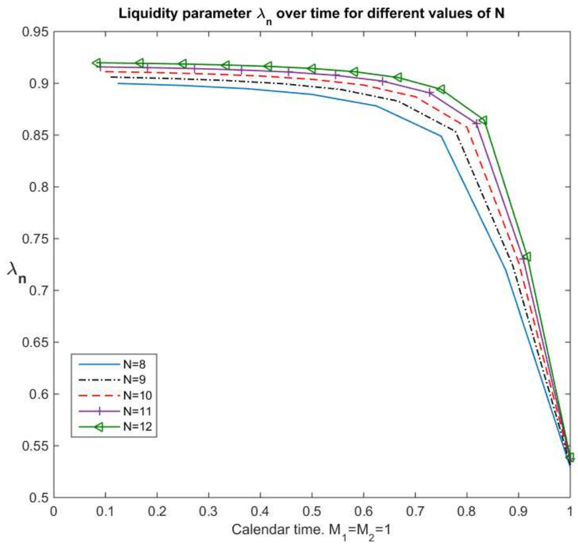

Figure 1: *Benchmark model. The figure compares the liquidity parameter for different HFT's speeds ($N = 8$, $N = 9$ $N = 10$ $N = 11$ and $N = 12$) as a function of time.*

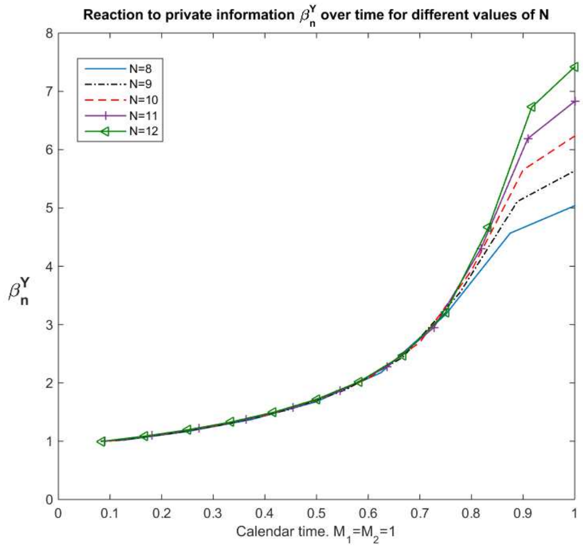

Figure 2: *Benchmark model.The figure compares the HFT's reaction to private information for different HFT's speeds ($N = 8$, $N = 9$ $N = 10$ $N = 11$ and $N = 12$) as a function of time.*

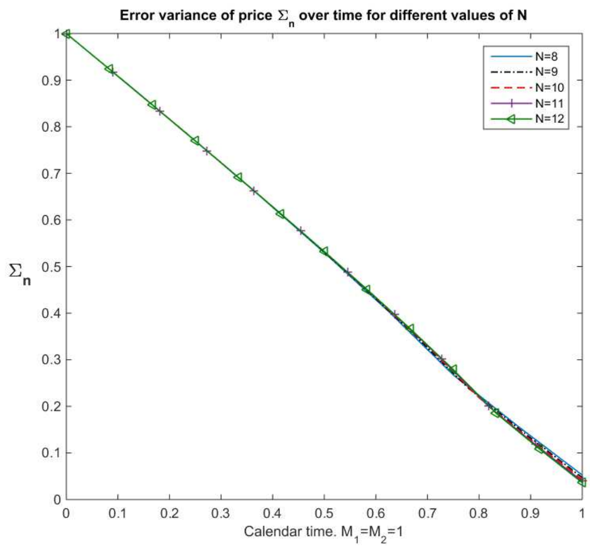

Figure 3: *Benchmark model. The figure compares price efficiency for different HFT's speeds* ($N = 8$, $N = 9$ $N = 10$ $N = 11$ *and* $N = 12$) *as a function of time.*

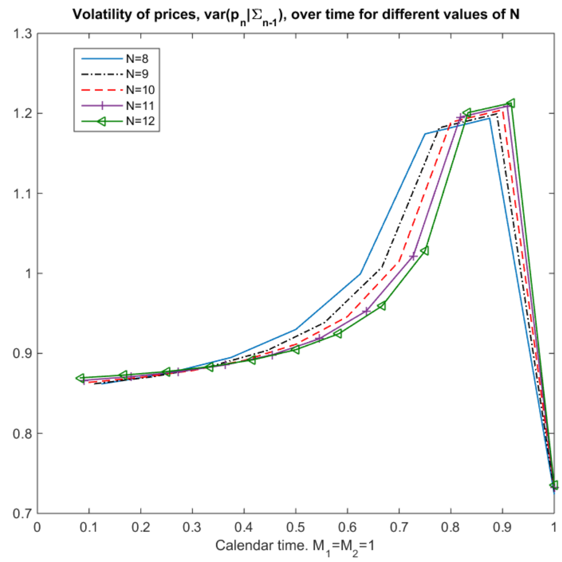

Figure 4: *Benchmark model. The figure compares price volatility for different HFT's speeds* ($N = 8$, 9, 10,11 *and* 12) *as a function of time. The number of HFTs and slow traders and equal to 1.*

9.2 Several HFTs and several slow traders

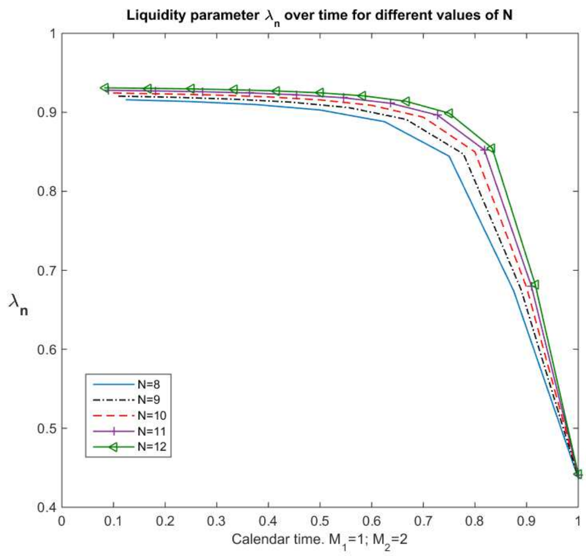

Figure 5: *General Model. The figure compares the liquidity parameter for different HFT's relative speeds ($N = 8$, $N = 9$, $N = 10$, $N = 11$ and $N = 12$) as a function of time. The number of traders is fixed at $M_1 = 1$, $M_2 = 2$.*

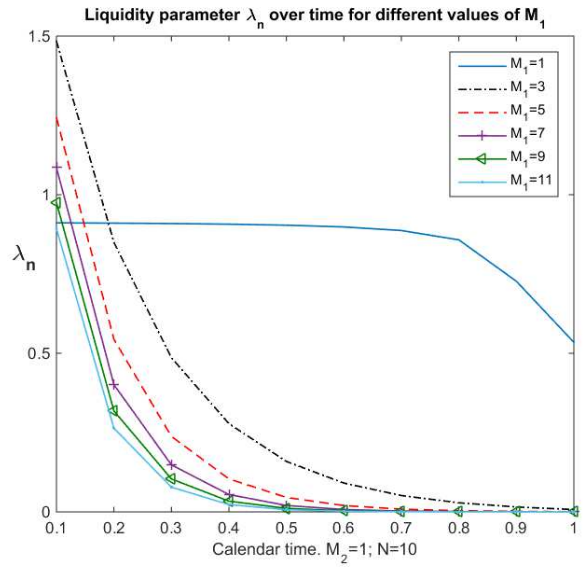

Figure 6: *General Model. The figure compares the liquidity parameter for different number of HFT. The number of slow traders is fixed at $M_2 = 1$ and the HFT's relative speed is set at $N = 10$.*

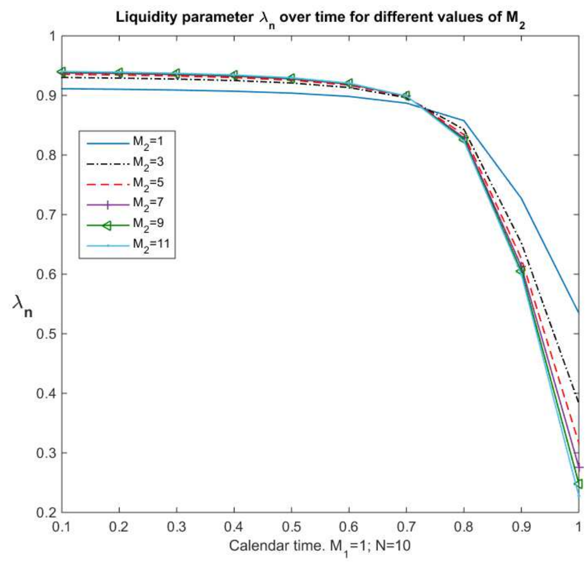

Figure 7: *General Model. The figure compares the liquidity parameter for different slow traders numbers ($M_2 = 1$, 3, 5, 7, 9 and 11) as a function of time. The HFT's speed is fixed at $N = 10$ and the number of HFTs is equal to 1.*

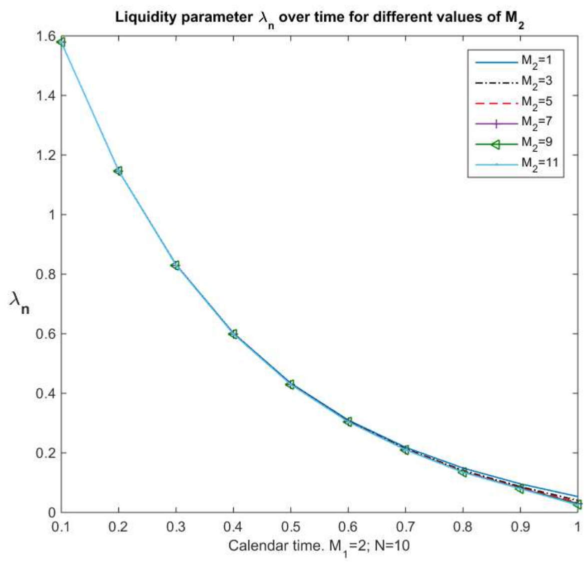

Figure 8: *General Model. The figure compares the liquidity parameter for different slow traders numbers ($M_2 = 1$, 3, 5, 7, 9 and 11) as a function of time. The HFT's speed is fixed at $N = 10$ and the number of HFTs is equal to 2.*

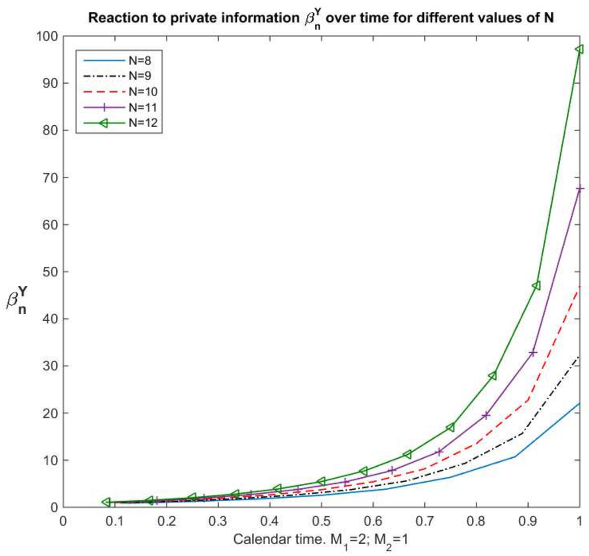

Figure 9: *General model. The figure compares the HFT's reaction to private information for different HFT's relative speeds ($N = 8$, $N = 9$, $N = 10$, $N = 11$ and $N = 12$) as a function of time. The number of HFTs is equal to* 1and the number of slow traders is equal to $M_2 = 2$.

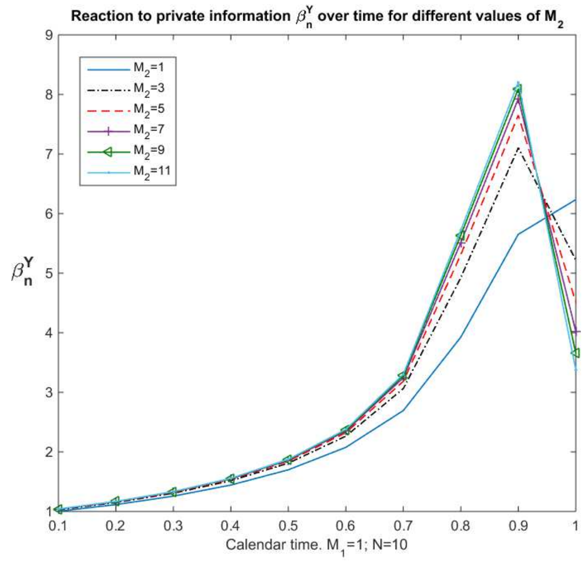

Figure 10: *General Model. The figure compares the HFT's reaction to private information for different slow traders numbers ($M_2 = 1$, 3, 5, 7, 9 and 11) as a function of time. The HFT's speed is fixed at $N = 10$ and the number of HFTs is equal to* 1.

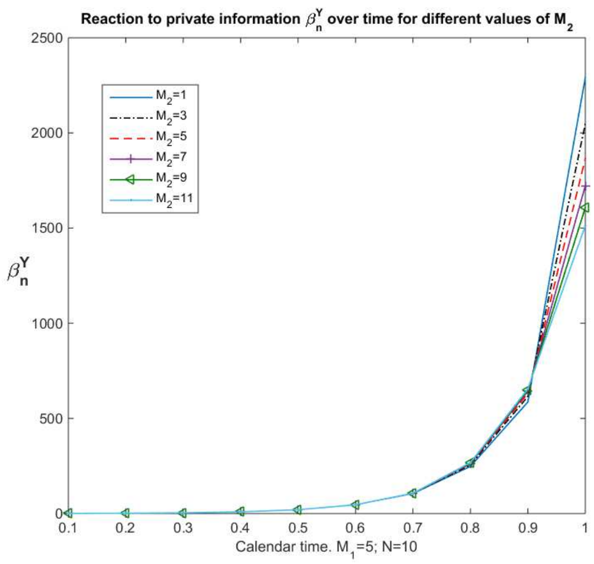

Figure 11: *General Model. The figure compares the HFTs' reaction to private information for different slow traders numbers ($M_2 = 1$, 3, 5, 7, 9 and 11) as a function of time. The HFT's speed is fixed at $N = 10$ and the number of HFTs is equal to* 5.

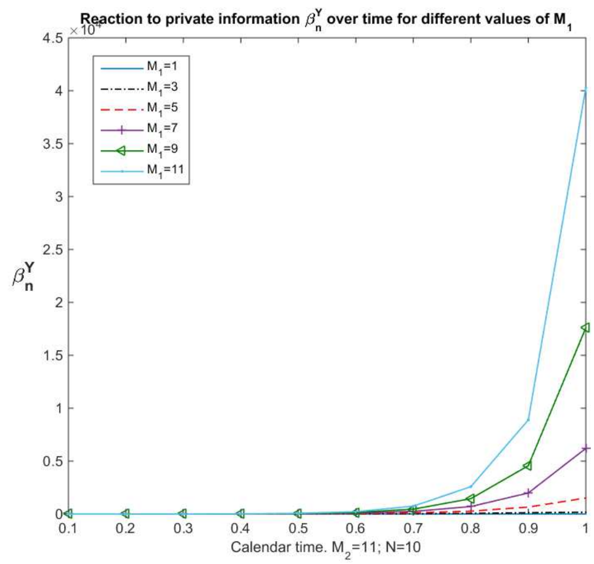

Figure 12: *General Model. The figure compares the HFTs' reaction to private information for different HFTs numbers ($M_1 = 1$, 3, 5, 7, 9 and 11) as a function of time. The HFT's speed is fixed at $N = 10$ and the number of slow traders is equal to* 11.

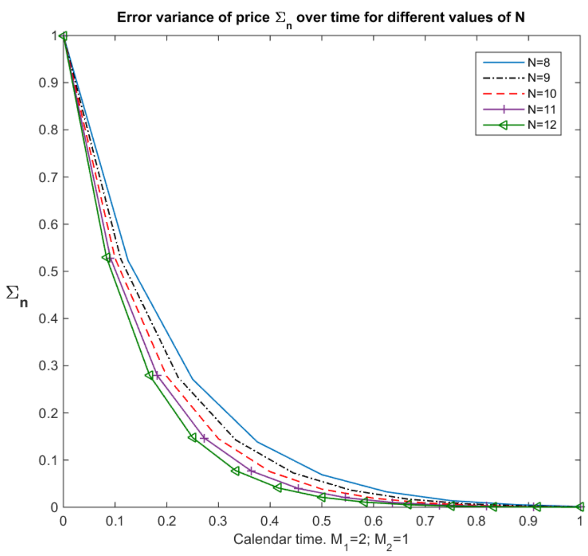

Figure 13: *General model. The figure compares price efficiency for different number of slow traders ($M_2 = 1, 3, 5, 7, 9$ and 11) as a function of time. The number of HFTs is set to* 1 *and the speed is set such that the fast trader can trade* 10 *times faster than the slow trader.*

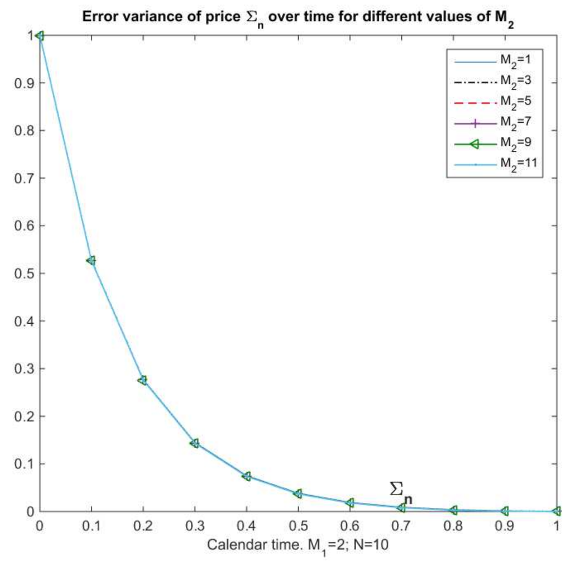

Figure 14: *General model. The figure compares price efficiency for different number of slow traders ($M_2 = 1, 3, 5, 7, 9$ and 11) as a function of time. The number of fast traders is set to* 2 *and the speed is set such that the fast trader can trade* 10 *times faster than the slow trader.*

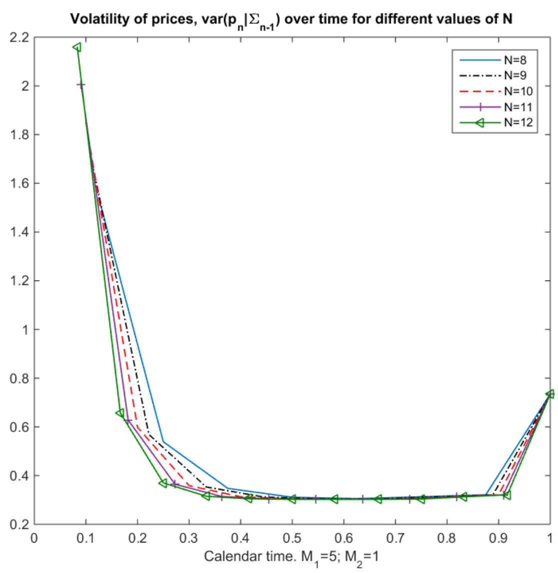

Figure 15: *General model. The figure compares price volatility for different HFT's speeds ($N = 8, 9, 10,11$ and 12) as a function of time. The number of fast traders is set to 5 and the number of slow traders is* 1.

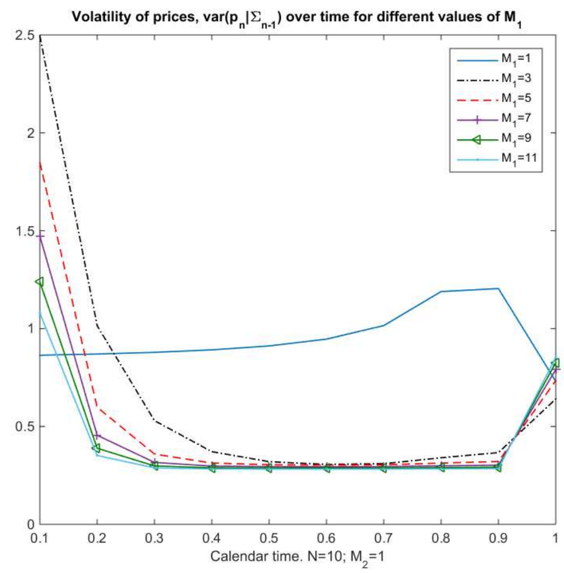

Figure 16: *General model. The figure compares price volatility for different number of HFTs ($M_1 = 1, 3, 5,7, 9$ and 11) as a function of time. The speed is set to* 10 *and the number of slow traders is* 1.

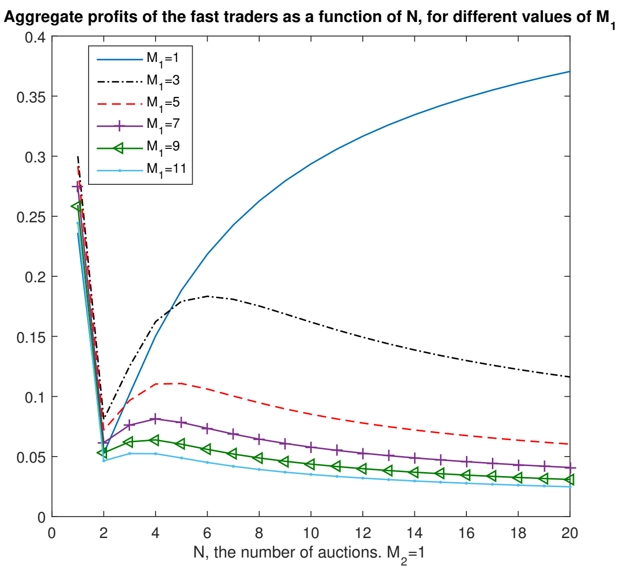

Figure 17: *General model. The figure compares the aggregate expected profits of the HFTs for different number of HFTs ($M_1 = 1, 3, 5, 7, 9$ and 11) as a function of their relative speed. The number of slow traders is set to* 1.

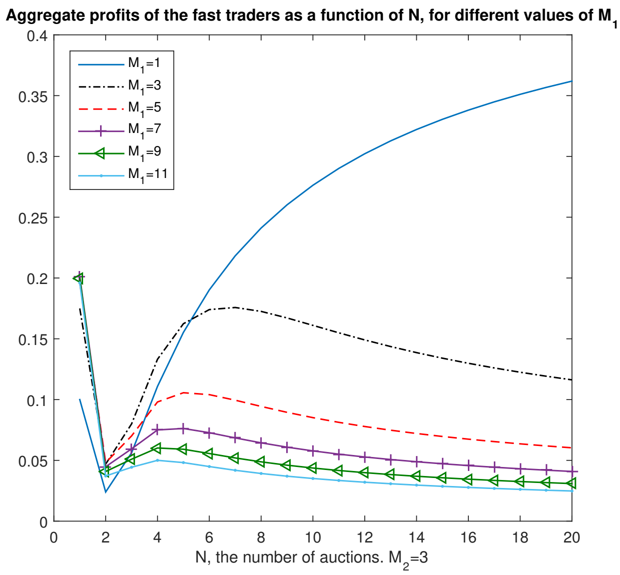

Figure 18: *General model. The figure compares the aggregate expected profits of the HFTs for different number of HFTs ($M_1 = 1, 3, 5, 7, 9$ and 11) as a function of their relative speed. The number of slow traders is set to* 3.

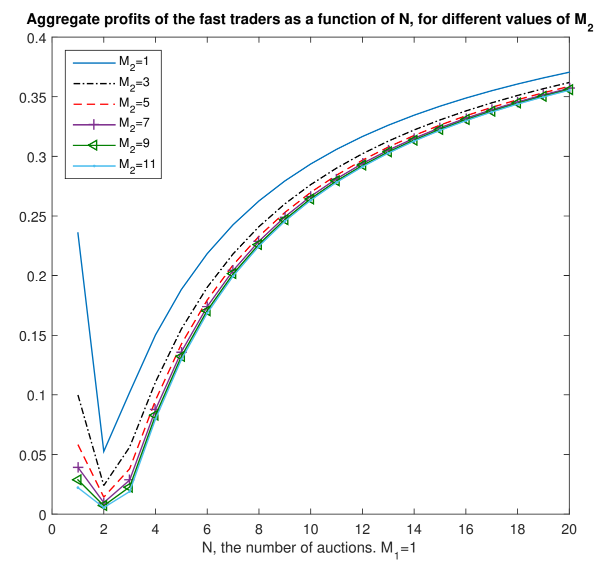

Figure 19: *General model. The figure compares the aggregate expected profits of the HFTs for different number of slow traders ($M_2 = 1, 3, 5, 7, 9$ and 11) as a function of their relative speed. The number of HFTs is set to* 1.

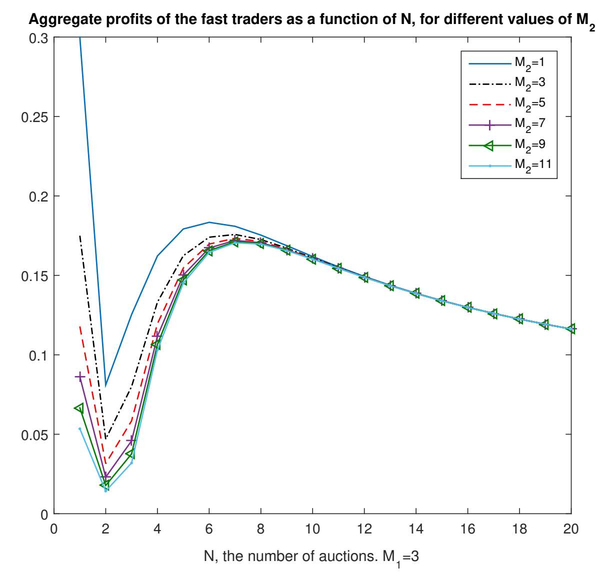

Figure 20: *General model. The figure compares the aggregate expected profits of the HFTs for different number of slow traders ($M_2 = 1, 3, 5, 7, 9$ and 11) as a function of their relative speed. The number of HFTs is set to* 3.

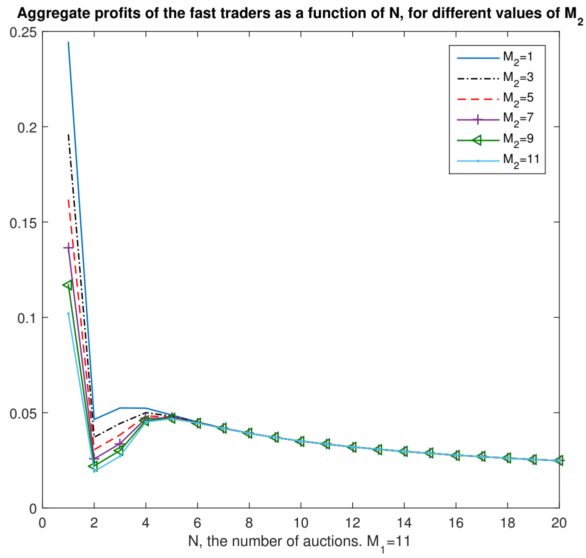

Figure 21: *General model. The figure compares the aggregate expected profits of the HFTs for different number of slow traders ($M_2 = 1$, 3, 5, 7, 9 and 11) as a function of their relative speed. The number of HFTs is set to* 11.

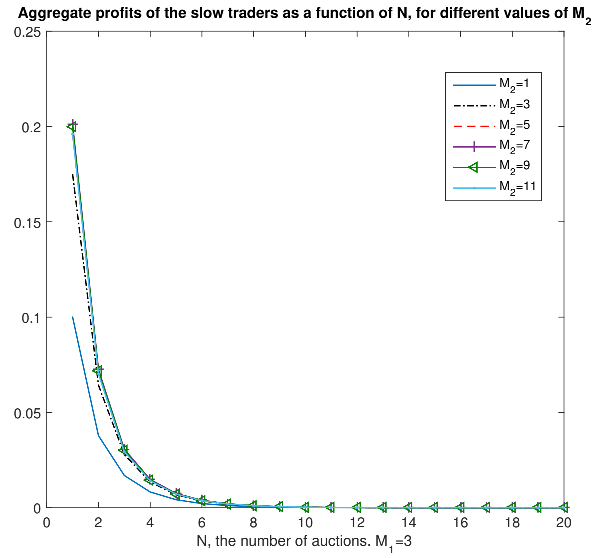

Figure 22: *General model. The figure compares the aggregate expected profits of the HFTs for different number of slow traders ($M_2 = 1$, 3, 5, 7, 9 and 11) as a function of their relative speed. The number of HFTs is set to* 3.

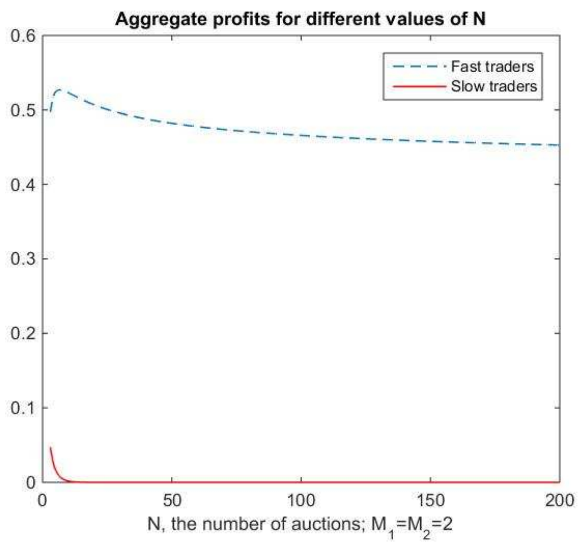

Figure 23: *General model. The figure compares the aggregate expected profits of the HFTs and the slwo traders for different HFTs' relative speed. The number of HFTs is set to* 2 *and the number of slow traders is set to* 2.