Modeling Technical Analysis

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Abstract

We present a model for the behaviour of a stock price process under the assumption of the existence of a support/resistance level, which is one of the most popular approaches in the field of technical analysis. We obtain optimal dynamic trading strategies to maximise expected discounted profit under the setup.

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1 Introduction

Many traders base their trading strategies on technical analysis (TA). The analysis uses heavily the visual shape of historical price graphs (which traders call 'charts') to determine whether the

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asset is a good buy or not. One of the basic modes of analysis in the field is that of a support and resistance line. In this method, the traders obtain a horizontal line called a support (resistance) line that they believe is a local support (roof) of the asset price. The belief is that if the stock price crosses a support line from above and goes lower than the level by a significant amount, then the stock has moved to a regime with negative outlook, in which case traders should sell, or, at least, not be long of the stock. On the other hand, if the asset price spikes up, crossing a resistance line from below, the asset is considered to have shifted to a regime with positive outlook and the traders should buy the asset or at least cover any short position.

We note here that the support/resistance level is not a hard limit. So, the stock can go lower (higher) than the support (resistance) level, without the regime changing and is expected to come back up (down) to the relevant side in a short period of time. We also note that there may be several support/resistance levels in one chart.

A level could, in theory, be a support level but not a resistance level and vice versa. However, since the level is where the regime changes it is natural to consider it to be both a support and resistance level. Compared to one regime, the other regime is relatively 'better' or 'worse'. Hence, if we are in the 'better' regime, the level which lies around the lower end of the regime is considered to be a (blurred) support line; if we are in the 'worse' regime, the same level which now lies around the upper end of the regime is considered to be a resistance line. When a support level becomes a resistance level or vice versa, we say that the stock has undergone a *regime transition*. This is in line with how traders think of the level.

Methods in TA are based on historical behaviour of stocks. They are not, as far as we know, currently supported by any economic theory, though they may be partially explained using behavioural science. Nevertheless, many traders believe they are useful and powerful. One reason is that the methods in TA are free from human emotions. Traders are consistently affected by the present performance of their portfolios and psychological stresses. Even if their trading instinct is sharp, the performance of their portfolios may deteriorate due to other non-trading factors. The decisions that TA makes are believed not to be affected by these factors.

The other reason why many traders support TA is that they believe in the strong form of the efficient market hypothesis (the stock price reflects not only information which is publicly available, but even the information that is not disclosed in public). So,for example, if an investor has insider information that potentially pushes the stock price lower, the trader might want to sell the stock before other people do to take advantage of possessing the information. The tradercan only extract benefit for himself by selling the stock in the market, which pushes the stock price lower. Even though the information is not publicly available, it is thus reflected in the price chart of the stock.

Remark 1. Instead of selling the stock directly in the market, an investor with insider information can seek other methods of benefiting himself from the expected stock performance. For example, the trader can buy a lot of naked puts on the stock. Then, the counterparty who sold the option to the investor has to sell the stock to hedge the position (unless the counterparty is happy holding it without any hedges). In either case, the investor with insider information will cause the market to sell the stock.

Studies on TA have been performed, but they mainly focus on how to detect the sign of the regime transition as quick as possible and checking against historical data (usually by means of computation and statistics) the usefulness of adopting TA in trading. Some examples of research that focus on these points are [2] and [13]. We know of no literature attempting to model and justify TA methods mathematically.

In order to model the price dynamics in the presence of support/resistance levels, we initially considered several possibilities. One is to use stochastic delay differential equations (SDDEs; [3], [14], [15], [24]). This makes sense as TA is the method traders use to forecast dynamics of the future stock price from analys of historical prices, and SDDEs are stochastic differential equations (SDEs) with coefficients that depend on historical levels. We rejected this method as overly-complicated for a first analysis.

The second method we considered was to model the stock price (at least locally near the support line) as a skew Brownian motion. Skew Brownian motion is described in [7]. Negative excursions from the origin of the standard Brownian motion are flipped with probability $1 - \alpha$ resulting in a process with Brownian dynamics away from the origin but with an upward bias. Using this process to describe the underlying stock price process under our setup requires less parameters than using SDDEs. However, as the papers of [18] and [22] show, the model with skew Brownian motion has arbitrage opportunities. As is pointed out in [18], we get a market which is arbitrage-free and complete within the class of simple strategies, but not in a more general context.

In this paper, we adopt a third approach. We assume that there are only two regimes in the stock price which follow different diffusion dynamics (which we will, from time to time, specialise to log-normal dynamics[1]). We then define criteria for buying/selling the stock by solving two optimal stopping problems: the first being the selling problem and the second the buying problem. Of course, we consider that purchase precedes selling temporally but this means, since we need to solve the problems iteratively, that we have first to solve the selling problem in order to specify the buying problem correctly.

[1] An earlier version of this paper appeared on arXiv and only considered the log-normal case.

One of the features that makes our model special is that the two regimes are not spatially distinct, i.e. there is a region where the stock price can be in either of the two regimes. This feature provides some "room" for the process in each regime to move around the support/resistence level without switching to the other regime.

The rest of the paper is organized as follows: Section 2 presents the setup we use for the model with a support/resistance level. We solve for the optimal stopping time (which is of stop-loss type) that maximizes the value function in Section 3. In Section 4, we solve the optimal stopping problem for purchasing the shares. We derive some conclusions in Section 5.

2 The price model and the selling problem

2.1 The model

We assume that there are price levels L and H $(0 < L < H)$ at which the regimes change. The positive regime corresponds to the interval $[L, \infty)$ and the negative regime to $(0, H]$. We may think of the support/resistance level as situated somewhere in (L, H), say at $\frac{L+H}{2}$.

The dynamics for the stock price are expressed in the following SDEs:

$$\begin{cases} dS_t^+ = \mu_+(S_t^+)dt + \sigma_+(S_t^+)dW_t & \text{in the positive regime,} \\ dS_t^- = \mu_-(S_t^-)dt + \sigma_-(S_t^-)dW_t & \text{in the negative regime,} \end{cases} \tag{2.1}$$

where σ_+, σ_-, μ_+, μ_- are Holder-continuous functions with σ_+ and σ_- strictly positive away from 0, and W_t is a one dimensional Brownian motion. Consistent with the modeling of a stock price, we assume that zero is either absorbing or inaccessible for a process following the negative-regime dynamics.We denote the associated differential operators (the restriction of the associated infinitesimal generators to $\mathbf{C}^2$ functions) by $\mathcal{L}^+$ and $\mathcal{L}^-$, so that

$$\mathcal{L}^+ : f \mapsto \frac{1}{2}\sigma_+^2 f'' + \mu_+ f' \text{ and } \mathcal{L}^- : f \mapsto \frac{1}{2}\sigma_-^2 f'' + \mu_- f'.$$

Let $r > 0$ denote the risk-free interest rate and we assume

$$\mu_- \leq r \leq \mu_+ \text{ and } r \geq 0. \tag{2.2}$$

We define the càdlàg flag-process F_t taking values in $\{-, +\}$ as

$$F_t = \begin{cases} + & \text{when the stock is in the positive regime} \\ - & \text{in the negative regime} \end{cases} . \tag{2.3}$$

Thus F_t jumps from one value to the other only in the following cases:

$$\begin{cases} F_{t-} = + \text{ and } S_t = L, \text{ then } F_t = - \\ F_{t-} = - \text{ and } S_t = H, \text{ then } F_t = + \end{cases} . \tag{2.4}$$

where the stock price satisfies

$$dS_t = \mu_{F_t}(S_t)dt + \sigma_{F_t}(S_t)dW_t. \tag{2.5}$$

Notice that the separation of L and H ensures that F only has finitely many jumps on any finite time-interval and that the pair (S_t, F_t) clearly is a Feller process.

We further assume that the trader always sells their shares at the level $M \geq H$, where the trader is happy to take profit or is required to do so by their manager.

Remark 2. It is not difficult to understand why a professional trader should take a profit at some level M, since the trader needs to choose a stock to invest in from a collection of stocks. If the trader wants to pick one from the group, not only does the trader compare the possible losses, but also the possible profits in investing in the stocks and will normally wish to close-out sufficiently profitable positions.

Note that the two regimes have non-empty intersection, $[L, H]$, of their domains. The condition (2.2) implies the discounted price process has supermartingale dynamics in the negative regime and submartingale dynamics in the positive regime.

2.2 Some notation and further assumptions

We denote generic exit/entrance times by τ so that

- the first entrance by S to the interval $[l, \infty)$ is denoted τ_l: $\tau_l = \inf\{t : \; S_t \geq l\}$
- the first entrance by S to the interval $[0, l]$ is denoted τ^l: $\tau^l = \inf\{t : \; S_t \leq l\}$
- similarly, the first entrance by S to the interval $[l, \infty)$, when the stock is in the negative regime is denoted τ_l^-: $\tau_l = \inf\{t : \; S_t \geq l \text{ and } F_t = -\}$
- the first exit by S from the interval (l, m) is denoted $\tau_{l,m}$: $\tau_l = \inf\{t : \; S_t \notin (l, m)\}$

We recall that there is a unique in law solution to the stock-price dynamics equations (2.1) and hence to (2.5) ([8]). Moreover, there are unique fundamental solutions on $\mathbb{R}_+$, which we denote ϕ_+, ψ_+, respectively ϕ_-, ψ_- to the ODEs $\mathcal{L}^+ f - rf = 0$, respectively $\mathcal{L}^- f - rf = 0$ satisfying $\phi_+(0) = \phi_-(0) = 0$; $\phi_+(H) = \phi_-(H) = 1$ and $\psi_+(L) = \psi_-(L) = 1$; $\lim_{x\to\infty}\psi_+(x) = \lim_{x\to\infty}\psi_-(x) = 0$ (see, for example, [5]).

2.3 The selling problem

If we assume that the trader already holds the stock, then if they wish to maximise their expected discounted profit they will seek the optimal time to sell. So they will seek a stopping time τ bounded by τ_M which achieves

$$\sup_{\tau\le\tau_M} \mathbb{E}[e^{-r\tau} S_\tau].$$

For each initial price $x \in (0, M]$ and flag-value $f \in \{+, -\}$ we define

$$\mathbf{V}(x, f) = \sup_{\tau\le\tau_M} \mathbb{E}_{x,f}[e^{-r\tau} S_\tau]. \tag{2.6}$$

If we suppose that $\mu_- \le \mu_+ \le r$ then the discounted stock price is actually a supermartingale bounded above by M and it follows immediately from characterisations of the solution to the optimal stopping problem that it is always optimal to sell the stock immediately. Conversely, if $r \le \mu_- \le \mu_+$ then the discounted stock price is actually a submartingale bounded above by M and now it is always optimal (by the optional sampling theorem) to wait until τ_M before selling. In the case outlined above, where the two drifts sandwich the risk-free rate, we expect the possibility of an earlier sale. As we shall see, the trader should never sell in the positive regime unless the stock price has attained level M (and earlier sale thus always corresponds to a "stop-loss" action). Defining

$$\tau_m^- \stackrel{def}{=} \inf\{t \ge 0 : \ F_t = - \text{ and } S_t \le m\}, \tag{2.7}$$

it follows that the optimal action is to sell at $\tau_M^{\hat{m}} \stackrel{def}{=} \tau_M \wedge \tau_{\hat{m}}^-$, the earlier of τ_M and $\tau_{\hat{m}}^-$,for some $\hat{m} < M$.

2.4 The form of the solution

Our analysis will divide into three cases:

(C1) where $\hat{m} \in [L, H]$,

(C2) where $\hat{m} \in (0, L)$ and

(C3) where $\hat{m} = 0$.

In case C1, notice that if $F_0 = +$ then the stock will be sold as soon as the price hits L. Recalling that the optimal future payoff (Snell envelope) for an optimal stopping problem is a martingale up until the (last) optimal stopping time, it follows that for such a value of $\hat{m}$, $e^{-r(t\wedge\tau_{L,M})}\mathbf{V}(S_{t\wedge\tau_{L,M}}, F_{t\wedge\tau_{L,M}})$ should be a martingale and

$$\mathbf{V}(L,+) = L \text{ and } \mathbf{V}(M,+) = M. \tag{2.8}$$

Similarly, if $F_0 = -$, $e^{-r(t\wedge\tau_H^{\hat{m}})}\mathbf{V}(S_{t\wedge\tau_H^{\hat{m}}}, F_{t\wedge\tau_H^{\hat{m}}})$ should be a martingale with

$$\mathbf{V}(H,-) = \mathbf{V}(H,+) \text{ and } \mathbf{V}(\hat{m},-) = \hat{m}, \tag{2.9}$$

for the optimal choice of $\hat{m}$.

In case C2, if $F_0 = +$,$e^{-r(t\wedge\tau_{L,M})}\mathbf{V}(S_{t\wedge\tau_{L,M}}, F_{t\wedge\tau_{L,M}})$ should again be a martingale, but now with boundary conditions

$$\mathbf{V}(L,+) = \mathbf{V}(L,-) \text{ and } \mathbf{V}(M,+) = M \tag{2.10}$$

and

$$\mathbf{V}(H,-) = \mathbf{V}(H,+) \text{ and } \mathbf{V}(\hat{m},-) = \hat{m}; \tag{2.11}$$

while in case C3 the requirements are the same as in case C2, with $\hat{m} = 0$, the boundary condition at 0 corresponding either to the inaccessibility of 0 or to the the fact that it is an absorbing boundary.

Standard arguments (see e.g. [8] etc.) tell us that the unique solution to the first characterisation is given by $\mathbb{V}(x, f)$ where $\mathbb{V}(x,+)$ satisfies

$$\mathcal{L}^+ v - rv = 0 \tag{2.12}$$

and with boundary conditions (2.8) and (then) $\mathbb{V}(x,-)$ satisfies

$$\mathcal{L}^- v - rv = 0 \tag{2.13}$$

with boundary conditions (2.9). In a similar fashion, by taking $\mathbb{V}(H,+) = \mathbb{E}_{H,+}[e^{-r\tau_{\hat{m},M}} S_{\tau_{\hat{m},M}}]$ and $\mathbb{V}(L,-) = \mathbb{E}_{L,-}[e^{-r\tau_{\hat{m},M}} S_{\tau_{m,M}}]$ and solving (2.12) and (2.13) with these values in (2.10) and (2.11) we obtain the unique solution to the second characterisation.

For the third case, the usual arguments show that, if we define ϕ_- by

$$\phi_-(x) = \mathbb{E}_{x,-}[e^{-r\tau_H}] \tag{2.14}$$

then ϕ_- satisfies (2.13) with boundary conditions $\phi_-(H) = 1$ and $\phi_-(0) = 0$.

Similarly, if we define $\phi^{L,M}$ and $\psi^{L,M}$ by

$$\phi^{L,M}(x) = \mathbb{E}_{x,+}[e^{-r\tau_M} 1_{(\tau_{L,M}=\tau_M)}] \text{ and } \psi^{L,M}(x) = \mathbb{E}_{x,+}[e^{-r\tau_M} 1_{(\tau_{L,M}<\tau_M)}],$$

then $\mathbb{V}(x,-) = \phi_-(x)\mathbb{V}(H,+)$ and

$$\mathbb{V}(H,+) = M\phi^{L,M}(H) + v(L,-)\psi^{L,M}(H),$$

so that

$$\mathbb{V}(H,+) = \frac{M\phi^{L,M}(H)}{1-\phi_-(L)\psi^{L,M}(H)}.$$

Consequently, if we take

$$\mathbb{V}(x,-) = \phi_-(x)\frac{M\phi^{L,M}(H)}{1-\phi_-(L)\psi^{L,M}(H)} \tag{2.15}$$

and $\mathbb{V}(x,+)$ as the unique solution to (2.12) with boundary conditions (2.10) we obtain the unique solution to the third characterisation.

2.5 Identifying the stop-loss boundary

To identify the stop-loss boundary we consider the argument we will make to show that our proposed solution is optimal. We will adopt the usual technique using the characterisation of the Snell envelope which works as follows: we will show that our candidate solution v has the following properties:

(P1) $\mathbb{V}_t \overset{def}{=} e^{-rt}v(S_{t\wedge\tau_M}, F_{t\wedge\tau_M})$ is a class D supermartingale;

(P2) $\mathbb{V}_t$ dominates the gains process $e^{-rt}S_{t\wedge\tau_M}$;

(P3) there exists a stopping time $\hat{\tau} \le \tau_M$ such that $\mathbb{V}_0 = \mathbb{E}[e^{-r\hat{\tau}}S_\tau]$.

This is sufficient to show that v is the optimal solution and that $\hat{\tau}$ is an optimal stopping time.

Now, to show this we'll characterise $\hat{m}$ as follows: in cases C1 and C2 we'll take $\hat{m}$ to be the unique choice of m for which

$$\frac{\partial v}{\partial x}|_{x=m} = 1, \tag{2.16}$$

while in the third case we'll show that $\frac{\partial v}{\partial x} \ge 1$ for all $x > 0$ (see Theorem 3.1).

We'll show that this is sufficient, since if we define $g(x,f) \overset{def}{=} v(x,f) - x$ then the strong maximum and strong minimum principles will tell us that $g \ge 0$, while smooth pasting at $\hat{m}$ (between v and the identity) gives us that $\mathbb{V}$ is a supermartingale, while it is clearly class D since bounded (by M).

Case C3 is similar except that we don't need smooth pasting when $\hat{m} = 0$.

Remark 3. It will be helpful in what follows to define the fundamental positive solutions to (2.12) and (2.13); we denote these by ϕ_+ and ψ_+ respectively ϕ_- and ψ_-. Thus these functions are all positive with ϕ_+ and ϕ_- increasing and with ψ_+ and ψ_- decreasing and we take $\phi_-(H) = 1 = \psi_+(L)$.

3 Establishing the solution of the selling problem

3.1 The general case

Theorem 3.1. *Under the model outlined in section 2 there are three possibilities corresponding to cases C1 to C3: either*

(S1) letting $v_1(x,f)$ be the solution to the linked ODEs (2.12) and (2.13) with boundary conditions (2.8) and (2.9); then there exists $\hat{m} \in [L,M]$ such that $\frac{\partial v_1}{\partial x}(x,-;m)|_{x=m=\hat{m}} = 1$ and then

$$v(x,f) = \begin{cases} v_1(x,f;\hat{m}) : \ x \in [\hat{m},M], f = - \ \text{or} \ x \in [L,M], f = + \\ x : \ x \in [0,\hat{m}), f = - \ \text{or} \ x \in [0,L], f = +; \end{cases} \tag{3.1}$$

or

(S2) for every $m \in [L,M]$, $\frac{\partial v_1}{\partial x}(x,-;m)|_{x=m} > 1$ but, defining $v_2(x,f;m)$ to be the solution to the odes (2.12) and (2.13) with boundary conditions (2.10) and (2.11), there exists an $\hat{m} \in (0,L)$ such that $\frac{\partial v_2}{\partial x}(x,-;m)|_{x=m=\hat{m}} = 1$ and then

$$v(x,f) = \begin{cases} v_2(x,f;\hat{m}) : \ x \in [\hat{m},M] \\ x : \ x \in [0,\hat{m}); \end{cases} \tag{3.2}$$

or

(S3) for every $m \in (0,L)$, $\frac{\partial v_2}{\partial x}(x,-;m)|_{x=m=} > 1$ in which case

$$v(x,f) = v_3(x,f) \tag{3.3}$$

where v_3 is the solution to the odes (2.12) and (2.13) with boundary conditions (2.10) and satisfying (2.15).

We outline the proof here, relegating some details to the appendix.

Proof. First we require

Lemma 3.2. *Suppose our candidate optimal value function, given by S1-S3, and denoted by v, satisfies properties P1-P3 then it is optimal*

(see Appendix A for the proof).

So we seek to prove that v has properties P1-P3. To prove P3: notice that each v_i is continuous on $[0, M]$ so bounded (actually bounded by M) and by the usual arguments is actually given by $\mathbb{E}_{x,f}[e^{-r\tau_M^{\hat{m}}})S_{\tau_M^{\hat{m}}}]$, so that $\hat{\tau} = \tau_M^{\hat{m}}$. To prove P1: since $\mathbb{V}$ is bounded it is definitely of class D. Since L is strictly less than H, we have, from the Itô-Tanaka formula,

$$d\mathbb{V}_t = 1_{(t<\tau_M)}e^{-rt}\Bigg[\Bigg(\quad -rv(S_t, F_t; \hat{m}) + \mathcal{L}^+ v(S_t, +)1_{(F_t=+)} + L^- v(S_t, +)1_{(F_t=-)}\Bigg)dt$$
$$+dM_t + \big(\tfrac{\partial v}{\partial x}(\hat{m}, -; \hat{m}) - \tfrac{\partial v}{\partial x}(\hat{m}-, -; \hat{m})\big)dl_t^{\hat{m}}\Bigg], \tag{3.4}$$

where M is a continuous local martingale, $l^{\hat{m}}$ is the local time of S at $\hat{m}$, and the last term disappears in case C3 since 0 is either absorbing or inaccessible. Now since $v(x, -; \hat{m}) = x$ for $x \le \hat{m}$ and since (in cases C1 and C2) we have imposed the condition that $\frac{\partial v}{\partial x}(\hat{m}, -; \hat{m}) = 1$ we see that the local time term in (3.4) disappears. Then, thanks to (2.12) and (2.13) the other bounded variation terms in (3.4) disappear when either $F_t = +$ or $F_t = -$ and $S_t \ge \hat{m}$ so we are left with

$$d\mathbb{V}_t = 1_{(t<\tau_M)}e^{-rt}\Big[(\mu_-(S_t) - rS_t)1_{(F_t=-, S_t<\hat{m})}dt + dM_t\Big], \tag{3.5}$$

so that $\mathbb{V}$ is a local supermartingale (since $\mu_-(x) \le rx$). Then, since v is bounded it follows that $\mathbb{V}$ is a supermartingale as required. Finally, to establish P2: we need only prove that $v \ge x$. This is Lemma A.1 in Appendix A. $\diamondsuit$

3.2 The lognormal case

In this subsection, we assume that the price dynamics are lognormal in each regime so that

$$\sigma_+(x) = \sigma_+ x; \ \sigma_-(x) = \sigma_- x; \ \mu_+(x) = \mu_+ x; \ \mu_-(x) = \mu_- x, \tag{3.6}$$

with

$$\sigma_+ > 0; \ \sigma_- > 0; \text{ and } \mu_- < r < \mu_+.$$

The corresponding fundamental solutions of (2.12) and (2.13) are given by x^{α_+}, x^{β_+} and x^{α_-}, x^{β_-} respectively, where α_+ and β_+ are the roots of $h_+(t) \stackrel{def}{=} \frac{1}{2}\sigma_+^2 t^2 + (\mu_+ - \frac{1}{2}\sigma_+^2)t - r = 0$ and α_- and β_- are the roots of $h_-(t) \stackrel{def}{=} \frac{1}{2}\sigma_-^2 t^2 + (\mu_- - \frac{1}{2}\sigma_-^2)t - r = 0$.

Lemma 3.3. *If the price dynamics correspond to (3.6) then case C3 does not occur.*

Proof. Notice that $h_-(0) < 0$ and $h_-(1) = \mu_- - r < 0$. Since $\sigma_-^2 > 0$ it follows that $\beta_- < 0 < 1 < \alpha_-$.

Consequently $\phi_-(x) = (\frac{x}{H})^{\alpha_-}$ and $v_3(x, -; 0) = cx^{\alpha_-}$ for some positive constant c. Thus $\frac{\partial v_3}{\partial x}(0, -; 0) = 0$ and so $\hat{m} > 0$.

$\diamondsuit$

Example 3.4. *Take $r = .02$, $\sigma_+^2 = .06$, $\mu_+ = .04$, $\sigma_-^2 = .01$ and $\mu_- = .005$. Then let $L = 1$, $H = 2$ and $M = 8$.*

The general solution to (2.13)is $Ax^{\frac{2}{3}} + Bx^{-1}$, so for $m \geq 1$, $v_1(x, +; m)$ satisfies $v_1(x, +) = Ax^{\frac{2}{3}} + Bx^{-1}$ with $A + B = 1$ and $4A - \frac{1}{8}B = 8$ so $A = \frac{65}{33}$ and $B = -\frac{32}{33}$. It follows that $\frac{\partial v_1}{\partial x}|_{x=m} > 1$ for all $m \in [L, M]$, and so $0 < \hat{m} < L = 1$ and $v(L, +) > L$.

4 Optimal Timing of Purchase

We now consider the optimal time to purchase the stock.

The gains function g is given by $g : (x, f) \mapsto v(x, f) - x$. Recalling that $g(x, f) = v(x, f) - x$, our gains process is $\mathbf{G}$, given by

$$\mathbf{G}_t = e^{-rt} g(S_t, F_t), \tag{4.1}$$

and we seek to find:

$$\mathbf{U}_t \stackrel{def}{=} \text{ess sup}_{\tau \geq t} \mathbb{E}[\mathbf{G}_\tau | \mathcal{F}_t].$$

The optimal stopping problem corresponds to the case when the buyer pays interest (or at least incurs a notional opportunity cost of interest foregone) on the purchase price from the time of purchase and seeks to maximise their profit.

Remark 4. There are other possibilities for the gains in the buying problem, such as proportional reward, where $g = v/x$. This corresponds to maximising profit per unit of expenditure.

Theorem 4.1. *It is optimal to purchase the shares only when the underlying process is in the positive regime. In this case, there is an optimal level $B \in [L, M)$, given by $B = \arg\max_{x \in [L,M]}[\frac{g(x,+)}{\psi_+(x)}]$, such that it is optimal to buy if and only if the stock has a price in $[L, B]$*

We give the proof in the Appendix

Example 4.2. *If we return to the set-up of Example 3.4, we see that $g(x, +) = Ax^{\frac{2}{3}} + Bx^{-1}$, where*

$$A + B = v(L, +) > L = 1, \; 4A - \frac{B}{8} = 8 \tag{4.2}$$

and $\psi_+(x) = x^{-1}$. *So* $\kappa = A + B - 1$ *and* $f_\kappa{}'(1) = \frac{2}{3}A + (\kappa - B) - 1 = \frac{5,3}{A} - 2$. *Now it follows from (4.2) that* $A > \frac{65}{33}$ *and so* $f_\kappa{}'(1) > 0$ *and thus* $B > L$.

5 Conclusions

We started with a simple setup where we only have one support/resistance level and fully showed the optimal level to sell the shares. We also considered the optimal timing to purchase the shares and found out that it is only optimal to do so in the positive regime when the investors borrow money to finance the share purchase.

Note that we rejected a model based on skew Brownian motion (with partial reflection at a resistance level), our solution could be used in this setting, at least if other dynamics remained the same: so we would assume partial reflection upwards at the level $R \in (L, M)$ when the stock is in the positive regime and partial reflection downwards at R when the stock was in the negative regime. The corresponding dynamics would have generators $\mathcal{L}^+$ and $\mathcal{L}^-$ with scale measures with a "kink" at R—upwards in the case of the positive regime and downwards for the negative regime (see[7] and [6]).

Example 5.1. *We take* $\sigma_+^2 = \sigma_-^2 = 0.06$, $r = 0.03$*); with* $L = \frac{3}{2}$, $H = \frac{5}{2}$ *and* $M = \sqrt{10}$. *The relevant scale functions are given (up to arbitrary constants) by*

$$s'_+(x) = \begin{cases} 9e^{-x}, \ x < 2, \\ 5e^{-x}, x > 2 \end{cases}$$

and

$$s'_-(x) = \begin{cases} 3e^{-x}, \ x < 2, \\ 8e^{-x}, x > 2 \end{cases}$$

The corresponding solution for the selling problem has $\hat{m} = 2$ *and (essentially) corresponds to case C1 with*

$$v(x, +) = \begin{cases} 2x - x^{-1}, \ x < 2, \\ \frac{3}{2}x + x^{-1}, x > 2 \end{cases}$$

and

$$v(x, -) = \frac{11}{6}x - \frac{10}{3}x^{-1} \text{ for } x \geq 2.$$

It is easy to check from first principles that the corresponding process $\mathbb{V}$ *satisfies conditions P1-P3 using the formulation that* $\mathcal{L}^f = \frac{d}{dm_f}\frac{d}{ds_f}$, *where* m_f *and* s_f *are the scale and speed measures for the relevant dynamics and hence that we do have the optimal solution.*

Intuitively, we can think of the discontinuity in s'_f at R as corresponding to there being an infinitesimal region around R where the drift μ_f is very large in modulus and thus case C1 allows for the derivative $v'(x,-;2)$ to jump from below 1 to above 1 at the kink

We do not seek to analyse this case further here.

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A

Proof of Lemma 3.2 Take the price process started at x, f and consider the corresponding process $\mathbb{V}_t$. By P1 and the Optional Sampling Theorem for Class D supermartingales, for any stopping time τ,

$$v(x,f) = \mathbb{V}_0 \geq \mathbb{E}_{x,f}[e^{-r\tau}v(S_\tau)] \geq \mathbb{E}_{x,f}[e^{-r\tau}S_\tau] \text{ (the last inequality follows by P2)}$$

and it follows that $v \geq V$. Conversely, by P3,

$$\mathbb{V}_0 = \mathbb{E}_{x,f}[e^{-r\hat{\tau}} S_{\hat{\tau}}]$$

and so $v \leq V$. $\square$

Lemma A.1. *The function* $\mathbb{V}(x,+) \geq x$ *for all* $x \in [L, M]$ *while* $\mathbb{V}(x,-) \geq x$ *for all* $x \in [0, H]$.

Proof. As indicates in section 2, the main tool here is the strong maximum/minimum principle (see [5] or [20]). First, define $g(x,f) \stackrel{def}{=} \mathbb{V}(x,f) - x$. Then

$$\mathcal{L}^+ g(x,+) - rg(x,+) = rx - \mu_+(x) \leq 0 \text{ on } [L, M].$$

with $g(M,+) = 0$ and $g(L,+) = g(L,-)$. Now the strong minimum principle tells us that $g(\cdot,+)$ has no negative minimum on (L, M), so to show that $g(\cdot,+)$ is non-negative on $[L, M]$ it is sufficient to show that $g(L,-) \geq 0$. This is immediate in case C1, since in this case $g(L,-) = 0$. It remains to show that

$$g(x,-) \geq 0 \text{ on } [\hat{m}, H]. \tag{A.1}$$

We now define ϕ by $\phi(x) = rx - \mu_-(x)$, define $g_\epsilon : [\hat{m}, H] \to \mathbb{R}$ by $g_\epsilon : x \mapsto g(x,-) + \epsilon x^{\frac{3}{2}}$, and note that $\mathcal{L}^- g_\epsilon - rg_\epsilon = \phi(x) + \epsilon\big(\frac{3}{8}\sigma_-^2(x)x^{-\frac{1}{2}} + \frac{1}{2}rx^{\frac{3}{2}} - \frac{3}{2}\phi(x)x^{\frac{1}{2}}\big) \geq (1 - \frac{3}{2}\epsilon x^{\frac{1}{2}})\phi(x)$. Thus, taking $0 < \epsilon < \frac{2}{3}M^{-\frac{1}{2}}$, $\mathcal{L}^- g_\epsilon - rg_\epsilon \geq 0$ on $(\hat{m}, H)$, so, by the strong maximum principle, g_ϵ has no positive maximum on $(\hat{m}, H)$. Moreover, $g_\epsilon(\hat{m}) = \epsilon \hat{m}^{\frac{3}{2}} > 0$ and $\frac{\partial g_\epsilon}{\partial x}|_{x=\hat{m}} = \epsilon \hat{m}^{\frac{1}{2}} > 0$ unless $\hat{m} = 0$ in which case $g_\epsilon(0) \geq 0$, $g_\epsilon{}'(0) = 0$ and $\frac{\partial^2 g_\epsilon}{\partial x^2}|_{x=\hat{m}} = \infty$. In either case, g_ϵ is initially *strictly* increasing and non-negative so must be monotone increasing on $[\hat{m}, H]$. We conclude that for each positive ϵ, g_ϵ is non-negative and monotone increasing on $[\hat{m}, H]$ and so, taking the limit as $\epsilon \to 0$ we conclude that $g(x,-)$ is increasing on $[\hat{m}, H]$. This establishes (A.1) $\diamondsuit$

It remains to establish our characterisation of $\hat{m}$.

Lemma A.2. $\hat{m}$ *as defined in Theorem 3.1 is well-defined.*

Proof. First, define ϕ_- as in (2.14); ψ_- as the unique (and decreasing) solution on $(0, H)$ of $\mathcal{L}^- f = 0$ with $\psi_-(H) = 0$ and $\psi_-(L) = 1$; ϕ_+ as the unique, increasing, solution of $\mathcal{L}^+ f = 0$ with $\phi_+(M) = 1$ and $\phi_+(L) = 0$; and ψ_+ as the unique, decreasing solution of $\mathcal{L}^+ f = 0$ with $\psi_+(M) = 0$ and $\psi_+(L) = 1$. Our solutions v_1, v_2 and v_3 to (2.12) and (2.13) will be of the form $\mathbb{V}(x, f; m) = A_f(m)\phi_f + B_f(m)\psi_f$ where, setting $C(m) = (A_-(m), B_-(m), A_+(m)B_+(m))'$, for suitable choices of the coefficients. It follows from the boundary conditions that

$$N(m)C(m) = (m, m - L, 0, M)' \tag{A.2}$$

in case S1 and

$$\tilde{N}(m)C(m) = (m, 0, 0, M)' \tag{A.3}$$

in cases S2 and S3, where

$$N(m) = \begin{pmatrix} \phi_-(m) & \psi_-(m) & 0 & 0 \\ \phi_-(m) & \psi_-(m) & \phi_+(L) & \psi_+(L) \\ \phi_-(H) & \psi_-(H) & \phi_+(H) & \psi_+(H) \\ 0 & 0 & \phi_+(M) & \psi_+(M) \end{pmatrix} \tag{A.4}$$

and

$$\tilde{N}(m) = \begin{pmatrix} \phi_-(m) & \psi_-(m) & 0 & 0 \\ \phi_-(L) & \psi_-(L) & \phi_+(L) & \psi_+(L) \\ \phi_-(H) & \psi_-(H) & \phi_+(H) & \psi_+(H) \\ 0 & 0 & \phi_+(M) & \psi_+(M) \end{pmatrix}. \tag{A.5}$$

It follows fairly easily from the Implicit Function Theorem that $\frac{\partial v}{\partial x}(x, -; m)$ is jointly continuous in (x, m).

Now take $L < m < H$, so that $\mathbb{V}(x, +; m) \geq x$ on $[L, M]$ and so $\mathbb{V}(m, -; m) = m$ and $\mathbb{V}(H, -; m) = \mathbb{V}(H, +; m) \geq H$. It follows that there is a $\theta \in (0, 1)$ such that $\frac{\partial v}{\partial x}(m + \theta(H - m), -; m) \geq 1$ and letting $m \uparrow H$ we see that $\frac{\partial v}{\partial x}(H, -; H) \geq 1$. Then the result follows by continuity. $\diamondsuit$

Proof of Theorem 4.1

We showed in the proof of Theorem 3.1 that

$$\mathcal{L}^- g(x, -) - rg(x, -) = (rx - \mu_-(x))1_{(\hat{m}, H)}(x) \geq 0,$$

and $g(x, -)$ is C^1 and piecewise C^2 on $(0, H)$. It follows that, defining τ_+ as the first time that the stock enters the positive regime, $\mathbf{G}_{t \wedge \tau_+}$ is a submartingale. Consequently, it is always optimal to continue (i.e. not purchase) when the stock is in the negative regime.

Conversely, fix $\kappa \geq 0$ and consider f_κ given by $f_\kappa : x \mapsto g(x, +) - \kappa \psi_+(x)$. Since ψ_+ satisfies (2.12) it follows that

$$\mathcal{L}^+ f_\kappa - r f_\kappa = rx - \mu_+(x) \leq 0,$$

so that f_κ satisfies the strong minimum principle on (L, M). In particular, defining ρ by setting $\rho = \frac{g(\cdot,+)}{\psi_+}$ and taking $\kappa = \rho(L)$ we see that $f_\kappa(L) = 0 \geq f_\kappa(M)$ and so either f_κ increases to a unique and strictly positive maximum in (L, M) or it is monotone decreasing on $[L, M]$. In either case we define B as the $\arg\max$ *and notice that in the first case, $\frac{\partial \rho}{\partial x}|_{x=B} = 0$ while in the second*

$\frac{\partial \rho}{\partial x}|_{x=B} \geq 0$. *Now define* u *by*

$$u(x,+) = \begin{cases} g(x,+) : & \text{if } x \leq B \\ \frac{\psi_+(x)}{\psi_+(B)} g(B,+) : & \text{if } x \geq B \end{cases}$$

and

$$u(x,-) = \phi_-(x) u(H,+).$$

This corresponds to the expected gain from buying at the time stated (since $\phi_-(x) = \mathbb{E}_{x,-}[e^{-r\tau_H}]$ *and* $\frac{\psi_+(x)}{\psi_+(B)} = \mathbb{E}_{x,+}[e^{-r\tilde{\tau}_B}]$*, where* $\tilde{\tau}_l \stackrel{def}{=} \inf\{t : S_t \leq l \text{ and } F_t = +\}$.

Thus, applying the arguments from the proof of Theorem 3.1, we see that we need only prove that

(1) $u \geq g$

(2) $\mathcal{L}^+ u(\cdot,+) - ru(\cdot,+) \leq 0$ *on* (L,B) *and*

(3) $u'(B) - g'(B) \leq 0$

We have already proved that $\mathcal{L}^+ v(\cdot,+) - rv(\cdot,+) \leq 0$ *on* (L,B) *in the proof of Theorem 3.1. Property (2) follows, since* $\mathcal{L}^+\psi_+ - r\psi_+ = 0$. *To establish Property (1), notice that* $u(\cdot,+) = g(\cdot,+)$ *on* $[L,B]$ *and* $[M,\infty)$. *On the interval* $[B,M]$, $u(x,+) - g(x,+) = \psi_+(x)\frac{g(B,+)}{\psi_+(B)} - g(x,+)$, *which is non-negative by the definition of* B.

Now define D *by* $D : x \mapsto u(x,-) - g(x,-)$. *Then, since* $\mathcal{L}^- D - rD = -(\mathcal{L}^- g(x,-) - rg(x,-)) = -(rx - \mu_-(x))1_{(\hat{m},H)}(x) \leq 0$, *it follows that* $e^{-r(t\wedge\tau_+)}(D(S_{t\wedge\tau_+})$ *is a bounded supermartingale with terminal value* $D(H)1_{\tau_H<\infty}$. *Since* $D(H) = u(H,-) - g(H,-) = u(H,+) - g(H,+)$, *and we have already shown that this is non-negative, it follows from the Optional Sampling Theorem that* $D(S_0) = u(S_0,-) - g(S_0,-) \geq 0$ *for each choice of* S_0 *in* $[0,H]$.

Property (3) follows from our characterisation of B, *since* $B > L$ *implies that* $\frac{\psi_+'(B)}{\psi_+(B)} = \frac{g'(B,+)}{g(B,+)}$, *while* $B = L$ *implies that* f_κ *is monotone decreasing on* $[L,M]$ *and so* ${f_\kappa}'(L) = g'(L,+) - u'(L,+) \leq 0$.

$\square$